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New results on inconsistency indices and their relationship with the quality of priority vector estimation

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ABSTRACT

The article is devoted to the problem of inconsistency in the pairwise comparisons based prioritization methodology. The issue of "inconsistency" in this context has gained much attention in recent years. The literature provides us with a number of different "inconsistency" indices suggested for measuring the *inconsistency* of the pairwise comparison matrix (PCM). The latter is understood as a deviation of the PCM from the *consistent case* – a notion that is formally defined in this theory. However the usage of the indices is justified only by some heuristics. It is still unclear what they *really* "measure". What is even more important and still not known is the relationship between their values and the "consistency" of the decision maker's judgments on the one hand, and the prioritization results upon the other.

In this paper we argue that it is necessary to distinguish between the three following tasks: the "measuring" of the "PCM inconsistency", the PCM-based "measuring" of the consistency of the decision maker's judgments and, finally, the "measuring" of the usefulness of the PCM as a source of information for estimation of the priority vector (PV). We present examples showing that improving the consistency of PCM may lead to poorer PV estimation results, and that such a situation may occur quite naturally. Next we focus on the third of the above tasks, which is very important one in multi-criteria decision making. For the first time in literature, with the help of Monte Carlo simulations, we analyze the performance of the most common inconsistency indices as indicators of the final PV estimates quality. We consider two types of PV estimation errors and examine their distributions as well as their relationship with the indices values. The new results presented here allow for a more profound interpretation of the well-known inconsistency characteristics. Moreover, based on the analysis, we also introduce a new inconsistency index. In comparison with the other ones, the new index manifests significantly higher correlation with PV estimation errors. This fact also enables us to propose a novel PCM acceptance approach that is supported by the classical statistical methodology.

1. Introduction: an issue of inconsistency in pairwise comparisons

One of the fundamental problems in decision making is the *prioritization* of available alternatives which is typically done by assigning a *priority weight* to each of them. The weights indicate the alternatives' relative importance with respect to a given criterion. The tuple of all priority weights forms a *priority vector* (PV) and deriving the PV on the basis of the information gathered from a decision maker (DM) the essence of all prioritization techniques. Many of these techniques are based on *pairwise comparisons* of the decision alternatives. As a result of such comparisons, a *pairwise comparison matrix* (PCM) is built – the elements of the PCM represent the DM judgments about the values of the priority weights' ratios. Although the idea of pairwise comparison is extremely natural and certainly very old, perhaps its first modern scientific applications were analyzed in Fechner (1860).

Nowadays the pairwise comparison is a common technique that is primarily used in the analytic hierarchy process (AHP) – one of the most popular tools for multi-criteria decision making (MCDM). AHP was developed in the seventies and eighties of the last century by Thomas Saaty. Saaty's seminal study (Saaty, 1977) had an undeniably great impact on the development of the pairwise comparisons based prioritization methodology. Present-day applications of the AHP include such diverse problems as aircraft evaluation (e.g. Bruno, Esposito, & Genovese, 2015), shipping management (e.g. Bulut, Duru, Keçeci, & Yoshida, 2012), sustainable biomass crop selection (e.g. Büyüktahtakın & Cobuloglu, 2015), supplier selection (Deng, Hu, Deng, & Mahadevan, 2014), evaluation of new service concepts (e.g. Lee, Lee, Seol, & Park, 2012), or some military tasks (Jin & Rothrock, 2010) to name just a few interesting examples from very recent years. Two theoretical issues connected with the usage of the pairwise comparisons are of special interest: the *choice of a prioritization technique*

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and *inconsistency evaluating*. The former refers to the PCM-based PV estimation methods, while the latter concerns "measuring" the credibility of the PCM (or of the DM her/himself) as a source of information about the PV. It is claimed (and it is quite intuitive) that serious errors in judgments about the priority ratios make the data contained in PCM useless and that they may result in poor PV estimates (see e.g. Saaty, 1980, 2004; Saaty & Vargas, 1984). In decision making practice it is a very important problem. Therefore, in recent years, we are presented with a number of papers dealing solely with the analysis of the inconsistency of the PCM. Such analysis has been considered as a "hot topic" in this field and "…the possibility of estimating inconsistency has been regarded as a valuable asset for techniques adopting pairwise comparison matrices" (Brunelli & Fedrizzi, 2015; Ishizaka & Labib, 2011). According to literature consistency control is nowadays "a unique and routine part of every AHP study" (Bulut et al., 2012), and the "possibility of evaluating, in a direct manner, the inconsistency of decision makers when eliciting the judgments" is of special importance in the AHP (Aguarón, Escobar, & Moreno-Jiménez, 2014; Altuzarra, Moreno-Jiménez, & Salvador, 2010). In Koczkodaj, Kułakowski, and Ligęza (2014) it is highlighted that "the expert judgment consistency should be evaluated and kept at a possible low level". The importance of the inconsistency measurement in the AHP practice was also emphasized in a number of application-oriented articles (e.g. Bruno et al., 2015; Bulut et al., 2012; Duru et al., 2012; Jin & Rothrock, 2010; Lee et al., 2012; Lin, Madu, Kuei, Tsai, & Wang, 2015; Rashvand, Majid, & Pinto, 2015) and/or in the context of group decision making (e.g. Aguarón et al., 2014; Brunelli & Fedrizzi, 2015; Lee et al., 2012; Lin et al., 2015; Zhang, Dong, & Xu, 2012). Yet, in Kułakowski (2015) the author examines the relationship between the inconsistency of the PCM and the so-called *condition of order preservation*, an intuitive requirement that was introduced in Banae Costa and Vansnick (2008).

One can also find a number of articles devoted to the development of procedures enabling the consistency "improvement" and/or "monitoring", usually with the underlying aim of improving the final estimate quality (see e.g. Benítez, Delgado-Galván, Izquierdo, & Pérez-García, 2012; Bozóki, Fülöp, & Koczkodaj, 2011; Koczkodaj & Szarek, 2010; Kou, Ergu, & Shang, 2014; Lamata, 2002; Liu, Zhang, & Zhang, 2014; Pereira & Costa, 2015; Saaty, 2003; Saaty & Ozdemir, 2003; Xia, Xu, & Chen, 2013).

In order to "measure" the inconsistency of a given PCM, various characteristics (called indices) are proposed. As a matter of fact these indices are not any measures (certainly not in the mathematical sense). They are just some kind of characteristics of the degree of the PCM deviation from the one obtained in a perfect judgment case. The first and perhaps still the most popular inconsistency characteristic is due to Saaty. In the fundamental paper (Saaty, 1977) he introduced an inconsistency index – denoted here as SI – which is closely related to his right eigenvalue prioritization method (REV). Another popular index is connected with a prioritization technique that is known as the row geometric mean method (GM). The GM was introduced in a paper (Crowford & Williams, 1985). In the same article the authors also suggested the *Geometric Consistency Index* (GI). The practical usage of this characteristic is analyzed e.g. in Aguarón and Moreno-Jiménez (2003). Yet another interesting proposal is due to Koczkodaj. In Koczkodaj (1993) he proposed an inconsistency index (KI) that is based upon the notions of a *triad* and its inconsistency. Koczkodaj's index KI is not connected with any specific prioritization technique. Its performance was analyzed in various papers (e.g. Bozóki & Rapcsák, 2008; Koczkodaj & Szwarc, 2014).

Apart from the indices SI, GI and KI, we are also presented with various other PCM inconsistency characteristics (e.g. Dijkstra, 2013; Grzybowski, 2010, 2012; Kazibudzki, 2012; Pelaez & Lamata, 2003 or Dong, Chen, Li, Hong, & Xu, 2015) for interval PCMs. There are also some proposals for measuring consistency in the fuzzy pairwise comparison framework, such as the centric consistency index (which is based on GI) proposed in Bulut et al. (2012). However it seems undoubtful, that these three above-mentioned indices (SI, GI and KI) are the most widely used ones in the pairwise comparisons methodology (see e.g. Brunelli & Fedrizzi, 2010, 2013, 2015; Choo & Wedlay, 2004; Dong, Xu, Li, & Dai, 2008; Dong et al., 2015; Grzybowski, 2012; Kou et al., 2014; Lin, 2007; Rashvand et al., 2015).

All the inconsistency indices known from literature have one common feature: they are nonnegative and they equal 0 only in the case of a perfectly *consistent* PCM – a notion formally defined in this theory. The users of these indices also *hope* that greater index values indicate worse consistency of the DM judgments. In some problems it would be perhaps the most desired property of any inconsistency index. However such a claim is supported only by some heuristic arguments. One can find articles where such arguments are based on various intuitive "psychological" requirements, which according to the authors' opinions, should be reflected by the index properties. Among the literature we can even find some interesting attempts to construct a system of intuitive, psychologically-justified axioms which should be satisfied by "good" inconsistency indices (Brunelli & Fedrizzi, 2013, 2015; Koczkodaj & Szwarc, 2014).

Another claim, fundamental for many applications, is the following: "the less consistent the DM judgments, the poorer are the PV estimates". It seems intuitive, but is it true? It turns out that it is not always – we provide examples here showing that the *improving* of the DM judgments *consistency* may lead to PV estimate's *errors increment*. Thus it is important to distinguish these two tasks:

- characterization of the dependence between the PCM and the consistence the DM judgments and
- characterization of the dependence between the PCM and the PV estimate's *errors*

To our best knowledge, all literature so far devoted to inconsistency analysis focuses on the first task or takes the existence of the desired dependencies (between index values, judgment consistency and magnitudes of PV estimation errors) for granted.

The first of the above tasks is certainly important in some situations (e.g. Brunelli & Fedrizzi, 2013, 2015; Temesi, 2011). For example it is argued that "… the more rational the judgments are, the more likely it is that the decision maker is a good expert with a deep insight into the problem …" (Brunelli & Fedrizzi, 2013). However this task is related to the psychological analysis of the decision making process and it is beyond the scope of this paper (although we will address some such issues very briefly).

In this article we focus on the second task, which is of primary interest in MCDM. We will study the relation between the values of inconsistency indices and the quality of the PV estimates (reflected in the magnitude of estimates errors). In this context we feel that the name "inconsistency index" should be replaced with "estimates quality indicator". However, in our paper, we study inconsistency characteristics that are already well known from literature, so we preserve the traditional terminology. Nonetheless, it should be understood that we are primarily interested in studying how to characterize the usefulness of the PCM as a source of information for estimation of the PV. The results of such studies allow the DMs deeper understanding of the information contained in the indices and may help her/him to choose the one that is good (or the best) in a given situation.

Finally we also have to address another terminological issue here. In literature one can come across the terms: *consistency* index and *inconsistency* index. Both can be found even in the same text and both are used even for the same characteristic. For example Saaty and Vargas (1984) use term "inconsistency" index, while later in Saaty (2004) and Saaty and Ozdemir (2003) the term "consistency index" is used (although the authors admit that de facto it indicates "inconsistency" of the PCM). In Deng et al. (2014) we can even read: "… a *consistency* index (C.I.) is defined to measure the *inconsistency* within the pairwise comparison matrix A". Dijkstra (2013) uses the terms