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A fast exact algorithm for the allocation of seats for the EU Parliament

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ABSTRACT

In this paper, we analyse the problem of allocation of seats for the EU Parliament. To solve it, we propose a fast exact algorithm which overwhelms limitations of the existing methods. It allows us to examine all feasible allocations of seats within few minutes. On this basis, an in-depth analysis of the problem is provided and some of its properties are revealed (e.g., the number of feasible allocations of seats holding the Treaty of Lisbon), which have never been presented in the scientific literature. Furthermore, the proposed algorithm is not limited to dealing with the problem of allocation of seats for the EU Parliament, but it can be applied in the expert system for any other similar problem, especially under degressive proportionality constraints.

1. Introduction

Undoubtedly, the allocation of seats for the European Parliament is not only a significant problem, but also a scientific challenge. Namely, the Committee on Constitutional Affairs (AFCO) of European Parliament commissioned a Symposium of Mathematicians to "*identify a mathematical formula for the distribution of seats which will be durable, transparent and impartial to politics*" (see Grimmett et al., 2011). Following Grimmett (2012), the purpose was to eliminate the political bartering which has characterised the distribution of seats by enabling a smooth reallocation of seats taking into account migration, demographic shifts and the accession of new Member States.

Let us recall the main documents. The Treaty of Lisbon (2010) constitutes that "*The European Parliament shall be composed of representatives of the Union's citizens. They shall not exceed* ***seven hundred and fifty*** *in number,* ***plus the President****. Representation of citizens shall be* ***degressively proportional****, with a minimum threshold of* ***six*** *members per Member State. No Member State shall be allocated more than* ***ninety-six*** *seats*".[1] Guidelines for understanding degressive proportionality can be found in the annex to the draft of the European Parliament resolution (Lamassoure & Severin, 2007). Furthermore, according to the same document "*the minimum and maximum numbers set by the Treaty must be fully utilized to ensure that the allocation of seats in the European Parliament reflects as closely as possible the range of populations of the Member States*". Thus, these documents outline requirements for feasible allocations of seats. To obtain an unprejudiced rule for the composition of the EU Parliament, a fair analysis is needed, which requires examining of all feasible allocations of seats (holding the above constraints). However, due to the intractability of the considered problem an exhaustive search cannot be applied. Therefore, lots of methods have been proposed, which construct compositions of the EU Parliament (e.g. Martínez-Aroza & Ramírez-González, 2008; Ramírez-González et al., 2012; Serafini, 2012; Słomczyński & Życzkowski, 2012). However, they face an essential problem – they are not able to generate (examine) all feasible allocations of seats (solutions). Thus, an algorithm that is able to find all feasible solutions is highly desirable.

In this paper, we will propose a fast exact algorithm LaRSA, which overwhelms boundaries of the existing methods and it allows us to examine all feasible allocations of seats for the EU Parliament in a reasonable time, which does not exceed few minutes. On this basis, we construct an expert system that allows us to provide an in-depth analysis, which has not been presented nor possible due to the limitations of the known methods. To the best of our knowledge the given properties of the composition of the EU Parliament (e.g., the number of feasible allocations of seats holding the Treaty of Lisbon) have never been presented in the scientific literature.

Note that the algorithm and the expert system presented in this paper, can be easily extended to analyse different allocation criteria (configurations). Furthermore, they are not limited to dealing with the problem of allocation of seats for the EU Parliament, but they can be applied for any other similar problem, especially under degressive proportionality constraints.

The remainder of this paper is organized as follows. In the next section, the allocation of seats is formulated as a combinatorial optimization problem. On this basis, the detailed description of the proposed algorithm is given, which is followed by a

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[1] Note that according to the newest decision of the EU Parliament (19 February 2013), there are 766 seats and 28 countries including Croatia.

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case study, where the algorithm is used to examine the allocation of seats for the EU Parliament. The last section concludes the paper.

2. Problem formulation

In this section, we will formally define the analysed problem. There are n countries, where p_i denotes the population of country i for $i = 1, \ldots, n$. For convenience, the countries are indexed according to the non-increasing order of p_i, i.e., $p_1 \leqslant p_2 \leqslant \cdots \leqslant p_n$, if it is not a case, we can renumber them. Each country i has assigned the number of seats s_i, where $s_i \in \{s_i^{\min}, \ldots, s_i^{\max}\}$ (for $i = 1, \ldots, n$) is the integer number, $s_i^{\min}$ and $s_i^{\max}$ are its minimal and maximal values, respectively. These values are globally bounded by the minimum m and the maximum M possible numbers of seats, i.e., $m \leqslant s_i^{\min}$ and $s_i^{\max} \leqslant M$ for $i = 1, \ldots, n$. The sum of all allocated seats (the house size) is H, i.e., $\sum_{i=1}^{n} s_i = H$. Due to the Treaty of Lisbon, allocations of seats are required to satisfy a condition of degressive proportionality, i.e., a sequence $s_1, s_2, \ldots, s_n$ is degressively proportional with respect to $p_1 \leqslant p_2 \leqslant \cdots \leqslant p_n$ if and only if $s_1 \leqslant s_2 \leqslant \cdots \leqslant s_n$ and $p_1/s_1 \leqslant p_2/s_2 \leqslant \cdots \leqslant p_n/s_n$. On this basis, the feasible allocation of seats (i.e., a solution) can be expressed as a tuple $S = (s_1, s_2, \ldots, s_n)$ of n elements (i.e., n-tuple), which has to hold the above constraints and Π is the set of all feasible solutions (allocation of seats), which hold the mentioned constraints.

For the actual case of the EU Parliament, there are millions of such feasible solutions (they are discussed further in Section 4). Therefore, in practice, they are evaluated in reference to some additional functions. Let $A_q: [0, +\infty) \to [0, +\infty)$ be a non-decreasing function with respect to the given population p, where rational values are allowed. It describes a desired ideal allocation of seats, i.e., "ideal quotas" (e.g. Ramírez-González et al., 2012; Serafini, 2012). On this basis, the objective is to find such a feasible solution $S \in \Pi$ that minimizes the criterion value $f_{A_q}(S)$ related with the function A_q. Formally, the optimal solution S^* is defined as follows $S^* \triangleq \arg\min_{S \in \Pi}\{f_{A_q}(S)\}$.

In the next section, we will present an exact algorithm that finds all feasible solutions Π for the known cases of the EU Parliament, whereas the calculations for particular A_q and f_{A_q} will be provided and analysed in Section 4.

3. The exact search space algorithm

To the best of our knowledge there are no efficient exact algorithms dedicated to the analysed problem of the allocation of seats for the EU Parliament nor to any other related problems. Therefore, in this section, we will describe the proposed algorithm that allows us to search the solution space and to find all feasible allocations of seats (solutions), i.e., the set Π. The searching process as well as Π are independent on A_q and f_{A_q}. In the further part, we will denote the proposed exact search space algorithm by LaRSA (Łyko and Rudek's Search Algorithm). Some preliminary concepts were presented in (Łyko et al., 2012).

Recall that according to the assumptions of feasible allocation of seats (resulted *inter alia* from the Treaty of Lisbon Lamassoure & Severin, 2007; Ramírez-González et al., 2012 or Słomczyński & Życzkowski, 2012), each feasible $S \in \Pi$ has to hold the following constraints:

C1: $s_1 = s_1^{\min} = s_1^{\max} = m$ and $s_n = s_n^{\min} = s_n^{\max} = M$,
C2: $m \leqslant s_i^{\min}$ and $s_i^{\max} \leqslant M$ for $i = 2, \ldots, n-1$,
C3: $s_1 \leqslant s_2 \leqslant \cdots \leqslant s_n$, where $s_i \in \{s_i^{\min}, s_i^{\max}\}$ for $i = 1, \ldots, n$
C4: $p_1/s_1 \leqslant p_2/s_2 \leqslant \cdots \leqslant p_n/s_n$,
C5: $\sum_{i=1}^{n} s_i = H$.

Let Π'' denote the solution space that contains all allocations, which hold assumptions C1–C3 and $|\Pi''|$ denotes its cardinality. The idea of the algorithm LaRSA is based on generating and searching the subset Π'' for finding solutions that hold C4 (degressive proportionality) and C5 (the total required number of seats). On this basis, the set Π of all feasible solutions is obtained and the optimal solution can be found.

At first, we will present the process of searching the solution space. It is based on generating in lexicographical order all possible tuples (allocations) that hold C1–C3. It is illustrated in Example 1.

Example 1. Let $n = 4$, $s_i^{\min} = m = 6$ and $s_i^{\max} = M = 8$ for $i = 1, \ldots, n$. According to C1–C3, the following tuples S are generated in lexicographical order: $(6,6,6,6)$, $(6,6,6,7)$, $(6,6,6,8)$, $(6,6,7,7)$, $(6,6,7,8)$, $(6,6,8,8)$, $(6,7,7,7)$, $(6,7,7,8)$, $(6,7,8,8)$, $(6,8,8,8)$, $(7,7,7,7)$, $(7,7,7,8)$, $(7,7,8,8)$, $(7,8,8,8)$, $(8,8,8,8)$.

Note that for example the solution $(6,8,7,6)$ is infeasible (according to C3), however, it can be reordered to $(6,6,7,8)$, which is feasible. On this basis, it can be observed in Example 1 that C1–C3 (especially C3) generate the 4-element multisets with elements from the 3-element set {6,7,8}, i.e., 4-combinations of 3 elements with repetitions in lexicographical order.

Let us extend the above observation on the considered problem. The number of the N-element multisets with elements from the K-element set is defined by the following binomial coefficients (multiset number):

$$\left(\!\!\binom{N}{K}\!\!\right) = \binom{N+K-1}{K} = \frac{N(N+1)(N+2)+\cdots+(N+K-1)}{K!}.$$

Since s_1 and s_n are fixed, then C1–C3 generate $N = n - 2$ multisets with elements from the set $\{m, \ldots, M\}$, thereby $K = M - m + 1$. For instance, if $n = 27$, $m = 6$ and $M = 96$, then we can calculate the cardinality of the solution space Π'' as follows (allocations that hold C1–C3):

$$|\Pi''| = \left(\!\!\binom{N}{K}\!\!\right) = \left(\!\!\binom{25}{91}\!\!\right) > 10^{25}.$$

In particular, if we assume that each solution (allocation of seats) can be generated and examined as one floating point operation and we are able to use the world's fastest supercomputer IBM Sequoila (TOP500, 2012), which performs 16.32 PFLOPS (i.e., 16.32×10^{15} floating point operations per second), then examining the solution space (for the given values of n, m, M) will take over 24 years.

Therefore, the proposed algorithm LaRSA do not generate the total set Π'', but at first it trims the range $\{s_i^{\min}, \ldots, s_i^{\max}\}$ for each s_i such that $s_i^{\min}$ (for $i = 2, \ldots, n-1$) are the greatest possible values (but not greater than M) that hold:

$$\frac{p_1}{s_1^{\min}} = \frac{p_1}{m} \leqslant \frac{p_2}{s_2^{\min}} \leqslant \cdots \leqslant \frac{p_{n-1}}{s_{n-1}^{\min}} \leqslant \frac{p_n}{s_n^{\min}} = \frac{p_n}{M},$$

whereas $s_i^{\max}$ (for $i = 1, \ldots, n$) are the smallest possible values (but not smaller than m) that hold:

$$\frac{p_1}{s_1^{\max}} = \frac{p_1}{m} \leqslant \frac{p_2}{s_2^{\max}} \leqslant \cdots \leqslant \frac{p_{n-1}}{s_{n-1}^{\max}} \leqslant \frac{p_n}{s_n^{\max}} = \frac{p_n}{M}.$$

Thus, we obtain the reduced set $\Pi' \subseteq \Pi''$. The feasible values $s_i^{\min}$ and $s_i^{\max}$ of seats for each country holding degressive proportionality for years 2007 and 2012 are presented in Table 1.

Although $\Pi' \subseteq \Pi''$, it is still intractable for the considered instances and an exhaustive search cannot be used. Therefore, the algorithm LaRSA searches the solution space Π' as a tree (see Fig. 1), which allows us to analyse not only complete solutions (allocations) but first and foremost partial solutions, which is fundamental for the proposed algorithm. Such an approach to