Expert Systems with Applications 39 (2012) 6967–6971

Contents lists available at SciVerse ScienceDirect

Expert Systems with Applications

journal homepage: www.elsevier.com/locate/eswa

Modelling ellipsoidal uncertainty by multidimensional fuzzy sets

Petr Hušek *

Department of Control Engineering, Faculty of Electrical Engineering, Czech Technical University in Prague, Technická 2, 166 27 Prague, Czech Republic

ARTICLE INFO

Keywords:
Ellipsoidal uncertainty
Modelling
Knowledge aggregation
Continuous-time systems

ABSTRACT

The paper addresses the problem of an aggregation of knowledge obtained from many sets of experiments each with different confidence level during identification of a linear dynamical system with uncertain parameters. The proposed approach describes the parameter space by a multidimensional fuzzy set specified by non-symmetric ellipsoidal α-cuts. An algorithm for determination of minimum confidence level guaranteeing stability of such a system is presented.

1. Introduction

When dealing with real systems it is not possible to obtain an accurate model of a system, some uncertainty has to be always considered. If the structure of a system is supposed to be given but the parameters are not known precisely we speak about parametric uncertainty. In engineering practice it is of fundamental importance that the systems preserve stable behaviour for a whole admissible parameter variations. In this view it could be also appropriate to know, if a system is stable for some nominal values of its parameters, within what boundary the stability remains preserved. Such a problem is called *stability margin* determination.

The problems mentioned above and solved by classical robust analysis approach Bhattacharyya, Chapellat, and Keel (1995) assume that the uncertainty remains the same independently on the working conditions. It means that the worst case has to be considered and conservative results are obtained. However, in many practical situations the uncertainty varies, e.g. depending on operation conditions. In such a case the uncertainty interval can be often parameterized by a confidence level. This parameter is usually tough to measure but it can be estimated by a human operator. If each coefficient of a system is described in this way the system corresponds to a family of interval linear time-invariant systems parameterized by the confidence level.

To handle such type of uncertain systems a mathematical framework is desired. Such a framework was proposed in Bondia and Picó (1999). They adopted the concept of *fuzzy numbers* and *fuzzy functions* (Dubois & Prade, 1980), their generalizations (Wei-xiang & Bang-yi, 2010) and fuzzy arithmetic (Hanss, 2005; Chen & Chen, 2009). The approach interprets a set of intervals parameterized by a confidence level as a fuzzy number with its membership degree given by this confidence level. It means that all the parameters c_i are characterized by means of fuzzy numbers, $\tilde{c}_i$, with membership functions $\alpha_i = \mu_{\tilde{c}_i}(\cdot)$. When a confidence level α_i is specified then the parameter interval is determined by the α_i-cut $[\tilde{c}_i]_{\alpha_i}$. If $\alpha_i = 1$ (the maximum confidence level – the system works in normal operating conditions) the parameter c_i can take any value (crisp or interval) within the cores of $\tilde{c}_i$'s ($c_i = \ker(\tilde{c}_i)$). If $\alpha_i = 0$ (the minimum confidence level) the parameter c_i is the interval equal to the support of $\tilde{c}_i$ ($c_i \in \mathrm{supp}(\tilde{c}_i)$).

Recently, various problems related to linear systems with fuzzy parametric uncertainty have been solved. In Bondia, Sala, Picó, and Sainz (2006) controller synthesis for such systems under fuzzy pole placement specifications is suggested. Several approaches of simulation of fuzzy discrete-time systems with uncertain initial state are proposed in Hanss (2002). A practical application of systems with parametric uncertainty characterized by fuzzy numbers is described in Seng, Nestorovic, and Vicini (2007).

The characteristic polynomial of a linear system with fuzzy parametric uncertainty with parameters entering its coefficients independently can be written as

$$\tilde{p}(s) = \tilde{a}_0 + \tilde{a}_1 s + \cdots + \tilde{a}_n s^n \quad (1)$$

where the coefficients $\tilde{a}_i,\ i = 0, \ldots, n$ are described by fuzzy sets with membership functions $\mu_{\tilde{a}_i}(\cdot)$.

For common confidence level $\alpha = \alpha_i$, the α-cut representation of polynomial (1) corresponds to an interval polynomial

$$[\tilde{p}(s)]_\alpha = a_0(\alpha) + a_1(\alpha)s + \cdots + a_n(\alpha)s^n \quad (2)$$

where $a_i(\alpha) = [\tilde{a}_i]_\alpha$.

The main task of stability analysis of such polynomial is to determine its *stability margin*, i.e. minimum confidence level α preserving stability of (2). The problem has been solved using a binary search in Nguyen and Kreinovich (1994) or using Argoun stability test (Argoun, 1987) in Bondia and Picó (2003) or with the help of Kharitonov theorem (Kharitonov, 1978) and Tsypkin–Polyak locus in Lan (2005).

* Tel.: +420 2 2435 7336; fax: +420 2 2491 8646.
E-mail address: husek@fel.cvut.cz

0957-4174/$ - see front matter © 2012 Elsevier Ltd. All rights reserved.
doi:10.1016/j.eswa.2012.01.021

Nevertheless, the parameters of a system or the coefficients of characteristic polynomials are very often identified using measured input–output data. In such case it is more realistic to characterize the set of parameters by a multidimensional membership function rather than employing fuzzy numbers. For example when utilizing well-known prediction error (PE) identification algorithm (Hanss, 1999; Söderström & Stoica, 1989; Bombois, 2000; Bombois, Gevers, Scorletti, & Anderson, 2001) the coefficients $\mathbf{a} = [a_0, \ldots, a_n]^T$ lie in an ellipsoidal set

$$(\mathbf{a}-\mathbf{a}^0)^T \mathbf{\Gamma}(\mathbf{a}-\mathbf{a}^0) \leqslant 1 \tag{3}$$

where $\mathbf{\Gamma}$ is a positive definite matrix and the vector $\mathbf{a}^0 = [a_0^0, \ldots, a_n^0]^T$ is the parameter nominal value. Performing more sets of measurements each with different confidence level α resulting in $\mathbf{\Gamma}(\alpha)$ it is reasonable, in order to aggregate the knowledge obtained from each of them, to characterize the coefficients by a fuzzy set described by the α-cuts

$$[A]_\alpha = \{\mathbf{a} : (\mathbf{a}-\mathbf{a}^0)^T \mathbf{\Gamma}(\alpha)(\mathbf{a}-\mathbf{a}^0) \leqslant 1\} \tag{4}$$

where the confidence level α indicates the belief in the experiment the measured data were obtained by. The natural question arises what minimum confidence level $\alpha_{\min}$ is necessary so that the α-cut polynomial (family of polynomials)

$$[\tilde{p}(s)]_{\alpha_{\min}} = a_0 + a_1 s + \cdots + a_n s^n; \quad \mathbf{a} \in [A]_{\alpha_{\min}} \tag{5}$$

remains stable.

2. Problem formulation

In the sequel we will consider polynomial $\tilde{p}(s)$ defined by its α-cut representation

$$\begin{aligned} [\tilde{p}(s)]_\alpha &= a_0 + a_1 s + \cdots + a_n s^n, \quad \mathbf{a} \in [A]_\alpha, \\ \mathbf{a} &= [a_0, \ldots, a_n]^T, \quad a_k \in \Re, \quad k = 0, \ldots, n \end{aligned} \tag{6}$$

with fuzzy set A characterized by the α-cuts

$$[A]_\alpha = \{\mathbf{a} : \mu_{\tilde{\mathbf{a}}}(\mathbf{a}) \geqslant \alpha\} = \{\mathbf{a} : (\mathbf{a}-\mathbf{a}^0)^T \mathbf{\Gamma}(\alpha,\mathbf{a})(\mathbf{a}-\mathbf{a}^0) \leqslant 1\} \tag{7}$$

where $\mathbf{a}^0 = [a_0^0, \ldots, a_n^0]^T$ is a nominal point and $\mathbf{\Gamma}(\alpha,\mathbf{a})$ is $(n+1) \times (n+1)$ square diagonal matrix

$$\mathbf{\Gamma}(\alpha,\mathbf{a}) = \begin{bmatrix} \frac{1}{\gamma_0^2(\alpha,a_0)} & & & 0 \\ & \frac{1}{\gamma_1^2(\alpha,a_1)} & & \\ & & \ddots & \\ 0 & & & \frac{1}{\gamma_n^2(\alpha,a_n)} \end{bmatrix} \tag{8}$$

with

$$\gamma_k(\alpha,a_k) = \begin{cases} \gamma_k^+(\alpha) & \text{for} \quad a_k \geqslant a_k^0 \\ \gamma_k^-(\alpha) & \text{for} \quad a_k < a_k^0, \quad k = 0, \ldots, n \end{cases} \tag{9}$$

where $\gamma_k^+(\alpha)$ and $\gamma_k^-(\alpha)$ are nonnegative decreasing functions defined for $0 \leqslant \alpha \leqslant 1$.

Let us note that for any $0 \leqslant \alpha \leqslant 1$ the corresponding coefficient space of the family of polynomials (6) is a non-symmetric axes-parallel hyperellipsoid with the lengths of the semiaxes given by $\gamma_k^-(\alpha)$ and $\gamma_k^+(\alpha)$ for coefficients lying below and above their nominal values, respectively. Non-symmetricity of the hyperellipsoid makes it possible to consider the cases when the nominal point corresponding to most often operating conditions does not lie in the middle of measured data.

Let us suppose that the most confident (1-cut) uncertain polynomial $[\tilde{p}(s)]_{\alpha=1}$ is robustly Hurwitz stable. The task is to find stability margin of the polynomial $\tilde{p}(s)$, i.e. minimum confidence level $\alpha_{\min} \in [0,1]$ such that uncertain polynomial $[\tilde{p}(s)]_\alpha$ is stable for $\alpha > \alpha_{\min}$ and unstable for $\alpha \leqslant \alpha_{\min}$.

In order to solve the problem a generalization of the Tsypkin–Polyak plot (Tsypkin & Polyak, 1991) will be used.

3. Generalized Tsypkin–Polyak plot

The main result of the paper is based on a modification of zero exclusion principle (Mansour, 1994).
Write the uncertain polynomial (6) as

$$\begin{aligned} [\tilde{p}(s)]_\alpha &= p(s,Q) = a_0 + a_1 s + \cdots + a_n s^n, \\ Q &= [A]_\alpha = \left\{ \mathbf{a} : \sum_{k=0}^{n} \left| \frac{a_k - a_k^0}{\gamma_k(\alpha,a_k)} \right|^2 \leqslant 1 \right\} \end{aligned} \tag{10}$$

and

$$p(j\omega,Q) = h(\omega,Q) + j\omega g(\omega,Q). \tag{11}$$

Denote $p_1(\omega,Q) = h(\omega,Q)/S(\omega) + jg(\omega,Q)/T(\omega)$ where $S(\omega)$ and $T(\omega)$ are positive functions of $\omega \geqslant 0$ such that $\lim_{\omega\to\infty} h(\omega)/S(\omega)$ and $\lim_{\omega\to\infty} g(\omega)/T(\omega)$ are finite.

Theorem 1 (Mansour (1994)). *The family of polynomials $p(s,Q)$ (10) with $[\tilde{p}(s)]_{\alpha=1}$ being stable is stable if and only if*

(a) *the coefficient a_n does not include 0,*
(b) *the coefficient a_0 does not include 0,*
(c) *$0 \notin p_1(\omega,Q)$ $\forall\omega > 0$.*

Let us again decompose a member of family of polynomials (10) into its even and odd part. For $s = j\omega$ we can write

$$p(j\omega,\mathbf{a}) = h(\omega,\mathbf{a}) + j\omega g(\omega,\mathbf{a}), \quad \mathbf{a} \in Q. \tag{12}$$

The nominal polynomial $p_0(s)$ evaluated at $s = j\omega$ then can be written as

$$p_0(j\omega) = p(j\omega,\mathbf{a}^0) = h_0(\omega) + j\omega g_0(\omega) \tag{13}$$

where

$$\begin{aligned} h_0(\omega) &= a_0^0 - a_2^0\omega^2 + a_4^0\omega^4 - \cdots, \\ g_0(\omega) &= a_1^0 - a_3^0\omega^2 + a_5^0\omega^4 - \cdots. \end{aligned} \tag{14}$$

Denote

$$\begin{aligned} S_2(\omega,\alpha) &= 0.5\left(S_2^-(\omega,\alpha) + S_2^+(\omega,\alpha)\right) + 0.5\left(S_2^-(\omega,\alpha) - S_2^+(\omega,\alpha)\right)\operatorname{sgn} h_0(\omega), \\ S_2^+(\omega,\alpha) &= \left(\sum_{k=0}^{n/4} \left(\gamma_{4k}^+(\alpha)\omega^{4k}\right)^2 + \sum_{k=0}^{(n-2)/4} \left(\gamma_{4k+2}^-(\alpha)\omega^{4k+2}\right)^2 \right)^{\frac{1}{2}}, \\ S_2^-(\omega,\alpha) &= \left(\sum_{k=0}^{n/4} \left(\gamma_{4k}^-(\alpha)\omega^{4k}\right)^2 + \sum_{k=0}^{(n-2)/4} \left(\gamma_{4k+2}^+(\alpha)\omega^{4k+2}\right)^2 \right)^{\frac{1}{2}}, \\ T_2(\omega,\alpha) &= 0.5\left(T_2^-(\omega,\alpha) + T_2^+(\omega,\alpha)\right) + 0.5\left(T_2^-(\omega,\alpha) - T_2^+(\omega,\alpha)\right)\operatorname{sgn} g_0(\omega), \\ T_2^+(\omega,\alpha) &= \left(\sum_{k=0}^{(n-1)/4} \left(\gamma_{4k+1}^+(\alpha)\omega^{4k}\right)^2 + \sum_{k=0}^{(n-3)/4} \left(\gamma_{4k+3}^-(\alpha)\omega^{4k+2}\right)^2 \right)^{\frac{1}{2}}, \\ T_2^-(\omega,\alpha) &= \left(\sum_{k=0}^{(n-1)/4} \left(\gamma_{4k+1}^-(\alpha)\omega^{4k}\right)^2 + \sum_{k=0}^{(n-3)/4} \left(\gamma_{4k+3}^+(\alpha)\omega^{4k+2}\right)^2 \right)^{\frac{1}{2}}. \end{aligned} \tag{15}$$

Without loss of generality suppose $a_n^0 > 0$. Then the key theorem can be stated.

Theorem 2. *Denote by α_∞ and α_0 the solutions of*

(a) $a_n^0 = \gamma_n^-(\alpha)$,
(b) $a_0^0 = \gamma_0^-(\alpha)$,*respectively, and by α_ω the solutions of*

(c) $\left(\frac{h_0(\omega)}{S_2(\omega,\alpha)}\right)^2 + \left(\frac{g_0(\omega)}{T_2(\omega,\alpha)}\right)^2 = 1$ *with respect to α for each $\omega > 0$, on the interval $\alpha \in [0,1]$. Assign zero in the case that a solution does not exist.Then*