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Improvement of grey models by least squares

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ABSTRACT

In this paper, we present an approach to the least squares solution to grey Verhulst model, and verify its feasibility by numerical examples. We also present the least squares solutions of grey models GM (1,1) and GM (2,1). For the convenience of applications in expert systems, the parameters computing formulas of grey models are also presented here. We carry out some numerical examples to examine the modeling precision of grey models in conventional way and in least squares. The numerical results reveal that the modeling precision of grey models in least squares is always better than that in conventional way.

1. Introduction

Time series prediction models are widely used in many areas, such as predicting stock market price indexes, foreign currency exchange rates (FX rates), population increasing rates, and so on. In attempts to understand the action of a system, observations are frequently made sequentially over time. Values in the future depend, usually in a stochastic manner, on the observations available at present. Such dependence makes it worthwhile to predict the future from its past. In other words, we will depict the underlying dynamics from which the observed data are generated, and will therefore forecast and possibly control future events. So, it seems essential to choice a reasonable mathematical model for time series processes and a reasonable solving approach. The ability to do prediction will influence the economic policy decision-making of large companies and governments, and the behavior by the financial actors.

Observations in financial area usually exhibit a distinguishing nonlinear feature. So, in some cases, it may be ill-suited to use linear time series models, such as AR models (AutoRegressive), MA models (Moving Average), ARMA models (Auto Regressive Moving Average), ARIMA models (Auto Regressive Integrated Moving Average). Nonparametric and parametric methods are the two main techniques for nonlinear time series modeling. Nonparametric methods include functional-coefficient autoregressive models (Chen & Tsay, 1993; Hastie & Tibshirani, 1993), adaptive functional-coefficient autoregressive models (Fan, Jiang, Zhang, & Zhou, 2003), additive models (Friedman & Stuetzle, 1981); while parametric methods include threshold models (Tong, 1990), ARCH and GARCH models (Gourieroux, 1997), bilinear models (Granger & Anderson, 1978).

Notwithstanding the applicableness and convenience of the methods mentioned above, for all modeling technique except grey modeling technique, invalidation is inevitable when the observed data available at present are very small. Stock market price indexes and FX rates are highly nonlinear, stochastic and highly non-stationary financial time series, and in some cases, the time series may exhibit a jumping phenomenon due to the sudden changing of the financial policy, and therefore, it is very difficult to fit a reasonable mathematic model to them by using all observed data in a time series. In such a case, it may be worthwhile to use some proximate observed data to model and obtain a sketch of the current development trend.

Grey modeling technique is well-known for limited observed data modeling. It was first introduced in early 1980s by Deng (1982). Since then, the grey system theory has become quite popular with its ability to deal with the systems on which we have only an incomplete information. As the superiority to conventional modeling technique, grey modeling techniques require only a limited amount of data to estimate the behavior of unknown systems (Deng, 1989). During the last two decades, grey system theory has been widely and successfully applied to various systems such as social and economic (Deng, 1985; Hsu, 2003; Hsu & Wang, 2009; Shen, Chung, & Chen, 2009), financial (Chang & Tsai, 2008; Chen, Chen, & Chen, 2008; Huang & Jane, 2009; Kayacan, Ulutas, & Kaynak, 2010; Wang, 2002), transportation (Mao & Chirwa, 2006), agricultural, mechanical, meteorological, ecological, hydrological, geological, medical, and military, systems.

Usually, grey models are adopted in light of the state of observed data distributing. In which, three grey models are frequently used in grey modeling techniques, they are grey model GM (1,1) which is suitable for the observed data with exponential

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distributing, GM (2,1) which is suitable for the observed data with oscillatory distributing, and grey Verhulst model which is suitable for the observed data with "S" distributing (i.e., saturated distributing). There are three existing approximations available for solving grey models mentioned above: ①taking the first datum in observed data sequence as the initial condition, such as the literature (Deng, 1985; Liu, Guo, & Dang, 1999; Xiao, Song, & Li, 2005); ②taking the last datum, such as the literature (Dang, Liu, & Liu, 2005; Li, Li, & Zhao, 1992); ③using least squares, such as the literature (Liu, Zhao, Zhai, Dang, & Zhang, 2003; Shen & Zhao, 2001; Xu & Leng, 1999). In fact, among the three approaches mentioned above, it is hard to say which one is the best. But in most cases, using least squares will make the error squares sum of fitting sequence to fitted sequence minimize, and hold the same principle as the one on which interim parameters of grey models are determined. However, there has been a lack of the least squares solutions to grey Verhulst model, and the problem is still where it was. So, it is worthwhile to study the least squares solutions to grey models. In the following work, we will present the least squares solutions of grey models GM (1,1), GM (2,1) and its parameters computing formulas. We will also devote to the derivation of the least squares solution of the grey Verhulst model, and verify its validity. Some numerical examples will be carried out to examine the modeling precision of grey models in conventional way and in least squares. The work presented here, the derivation of the least squares solutions of grey Verhulst model and its parameters computing formulas, makes a significant and new contribution to the grey modeling technique.

2. Grey model GM (1,1)

Suppose the primitive data sequence to be $X^{(0)} = (x^{(0)}(1), x^{(0)}(2), \ldots, x^{(0)}(n))$. According to grey system theory, we may have

$$1 - AGOX^{(0)} : X^{(1)} = (x^{(1)}(1), x^{(1)}(2), \ldots, x^{(1)}(n)) \tag{1}$$

and

$$MEANX^{(1)} : Z^{(1)} = (z^{(1)}(2), z^{(1)}(3), \ldots, z^{(1)}(n)), \tag{2}$$

in which the symbol $1 - AGOX^{(0)}$denotes the first order accumulating generation operator for $X^{(0)}$, i.e., $x^{(1)}(k) = \sum_{i=1}^{k} x^{(0)}(i),\ k = 1, 2, \ldots, n,\ n \geqslant 4$; and the symbol $MEANX^{(1)}$ denotes the mean consecutive neighbors generation operator for $X^{(1)}$,i.e., $z^{(1)}(k) = 0.5x^{(1)}(k) + 0.5x^{(1)}(k-1)$ ($k = 2, 3, \ldots, n$ $n \geqslant 4$). The grey model GM (1,1) may be written as

$$x^{(0)}(k) + az^{(1)}(k) = b \tag{3}$$

and its whitenization equation is

$$\frac{dx^{(1)}}{dt} + ax^{(1)} = b, \tag{4}$$

in which, a and b are the interim parameters. The parameter matrixes are

$$\hat{a} = \begin{bmatrix} a \\ b \end{bmatrix} = (B^T B)^{-1} B^T Y, \tag{5}$$

in which,

$$B = \begin{bmatrix} -z^{(1)}(2) & 1 \\ -z^{(1)}(3) & 1 \\ \vdots & \vdots \\ -z^{(1)}(n) & 1 \end{bmatrix}, \quad Y = \begin{bmatrix} x^{(0)}(2) \\ x^{(0)}(3) \\ \vdots \\ x^{(0)}(n) \end{bmatrix}.$$

The formulas of parameters are

$$\begin{cases} a = \frac{CD-(n-1)E}{(n-1)F-C^2} \\ b = \frac{DF-CE}{(n-1)F-C^2} \end{cases}, \tag{6}$$

in which, $C = \sum_{k=2}^{n} z^{(1)}(k),\ D = \sum_{k=2}^{n} x^{(0)}(k),\ E = \sum_{k=2}^{n} x^{(0)}(k) z^{(1)}(k),$ $F = \sum_{k=2}^{n} (z^{(1)}(k))^2$. The solutions of Eq. (4), i.e., the time response functions, are

$$x^{(1)}(t) = ce^{-at} + \frac{b}{a}. \tag{7}$$

Hence, the time response sequences of Eq. (3) are

$$\hat{x}^{(1)}(k) = ce^{-ak} + \frac{b}{a}; \quad k = 1, 2, \ldots, n, \tag{8}$$

in which, c is the undetermined integration constant. The condition to determine c is ①taking $x^{(1)}(1)$, ②taking $x^{(1)}(n)$, ③using least squares, respectively, then the computing formulas of c is, respectively

$$c = \left(x^{(1)}(1) - \frac{b}{a}\right)e^{a}, \tag{9}$$

$$c = \left(x^{(1)}(n) - \frac{b}{a}\right)e^{an} \tag{10}$$

and

$$c = \sum_{k=1}^{n} \left[x^{(1)}(k) - \frac{b}{a}\right] e^{-ak} \Bigg/ \sum_{k=1}^{n} e^{-2ak}. \tag{11}$$

After inversely accumulating generation operator from Eq. (8), we may obtain the simulation function of $X^{(0)}$ as follows

$$\hat{x}^{(0)}(k+1) = \hat{x}^{(1)}(k+1) - \hat{x}^{(1)}(k) = c(e^{-a} - 1)e^{-ak}. \tag{12}$$

The following example is intended to compare the modeling precision among three different conditions to determine c, mentioned above. We introduce a data set that is produced by $x^{(0)}(k+1) = e^{0.2k}$, so that we may obtain a primitive data sequence $X^{(0)} = (x^{(0)}(1), x^{(0)}(2), \ldots, x^{(0)}(n))$. Then, in light of grey system theory, we may carry out the calculation of time series modeling by GM (1,1). All results are listed in Tables 1–3, in which Case 1# is the case of taking $x^{(1)}(1)$ as the initial condition; Case 2# is the case

Table 1
The results while $X^{(0)}$ takes five elements.

k	$x^{(0)}(k)$	Case 1#		Case 2#		Case 3#	
		$\hat{x}^{(0)}(k)$	Δ_k	$\hat{x}^{(0)}(k)$	Δ_k	$\hat{x}^{(0)}(k)$	Δ_k
1	1.221403	1.221403	0	1.221403	0	1.221403	0
2	1.491825	1.486362	0.003662	1.508235	0.011	1.501381	0.006406
3	1.822119	1.814241	0.004323	1.819066	0.001675	1.817554	0.002505
4	2.225541	2.214448	0.004984	2.220338	0.002338	2.218492	0.003167
5	2.718282	2.702938	0.005645	2.710127	0.003	2.707874	0.003829
		$\Delta' = \frac{1}{5}\sum_{k=1}^{5}\Delta_k = 0.37\%$		$\Delta'' = \frac{1}{5}\sum_{k=1}^{5}\Delta_k = 0.36\%$		$\Delta''' = \frac{1}{5}\sum_{k=1}^{5}\Delta_k = 0.32\%$	