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Computing shadow prices/costs of degenerate LP problems with reduced simplex tables

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ABSTRACT

In applications of linear programming, shadow prices/costs are as important as the optimal values of decision variables and objective function. When the linear programming problems are primal degenerate, the shadow prices are no longer necessarily equal to optimal value of dual variables. In such cases, the so-called two-sided shadow prices are defined. However, existing approaches for two-sided shadow prices are tedious and unnoticed by decision-makers. The situation for shadow costs in a dual degenerate LP problem is the same. This study will first review and generalize the approaches of shadow prices in related studies, and then propose an easy approach to compute the shadow prices with respect to a resource and/or a resource bundle by using a reduced optimal simplex table. With slight modification, the proposed approach can also find the shadow costs with respect to an activity and/or an activity bundle. Numerical examples are used to illustrate the new approaches. Furthermore, some other important topics are discussed, as: the paradoxical situation, complementary effect between resources, weakly redundant constraint and the optimal change vector. We believe the proposed approaches are efficient and useful not only in classroom teaching but also in software programming of commercial packages.

1. Introduction

Numerous applications of linear programming (LP) exist in today's competitive business environment. Such applications solve practical small and large problems (Arsham, Adlakha, & Lev, 2009). LP also plays as an important role of problem analyzing (Higle & Wallace, 2003). Postoptimal analysis is the analyzing tool of LP. An important and useful issue related to postoptimal analysis is *shadow price* (or *imputed cost*). In many applications, such as in production, economics, finance *etc.*, the shadow price is even more important than the solution of the problem (Aucamp & Steinberg, 1982). Technically, each shadow price is defined as the rate of change of the optimal value function with respect to the change of the amount of one resource. In most textbooks on LP, shadow price is identical to the optimal dual value (Hillier & Lieberman, 2004, chap. 4). However, in the case an optimal basic solution is primal degenerate, occurs frequently in practice (Pan, 1998), there may be alternative optimal dual values. Therefore, shadow price is no longer necessarily equal to optimal dual value. Thus, their equivalence holds only under the assumption of primal non-degeneracy (Ho, 2000). Practice shows that an accurate concept is not widely known among LP practitioners, and most commercial software packages fail to provide correct information about shadow price (Koltai & Terlaky, 2000; Jansen, Jong, Roos, & Terlaky, 1997). Fallacious interpretations of shadow price can lead to expensive mistakes (Rubin & Wagner, 1990). In fact, in primal degenerate cases, so-called two-sided shadow prices should be defined (Gal, 1986, 1992), *i.e.*, the *negative* (*or selling*) *shadow price* and *positive* (*or buying*) *shadow price*. While the optimal dual value is not always unique; the two-sided shadow prices are always determined uniquely (Akgül, 1984; Stallaert, 2007).

The above definition refers to the so called traditional shadow price, in which only one resource changes at a time. For a *shadow price of a resource bundle* (*i.e.*, combination of resources) the amounts of several resources are allowed to change simultaneously (Akgül, 1984; Stallaert, 2007). This extension is of practical value when *the complementary effect between resources* occurs. The complementary effect between resources is a unique phenomenon of degenerate problems (Stallaert, 2007). When the effect occurs, shadow price of a resource bundle is worth more than the weighted sum of corresponding traditional shadow prices. In cases of primal degenerate, the two-sided shadow prices are also necessary for the shadow price of a resource bundle.

Similarly, *shadow cost* (or *reduced cost*) is defined as the rate of change of the optimal value function with respect to the change of the amount of one profit coefficient. *Shadow cost of activity bundle* is another important issue of post-optimal analysis. Two-sided shadow costs and shadow costs of an activity bundle are necessary when an optimal basic solution is dual degenerate. Shadow cost is

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as important as shadow price, whereas discussions of true shadow cost are less common than those of shadow price.

Degeneracy of optimal solutions causes considerable difficulties in post-optimal analysis (Hadigheh & Terlaky, 2006), including the determination of true shadow prices/costs. This study presents easy approaches to calculate shadow price, shadow price of resource bundle, shadow cost and shadow cost of activity bundle. The second section of this study does not only review the methods for true shadow price presented in previous research but also extends them to general form which computes shadow cost. In addition, the constraints of LP problems are no longer restricted to the ($\leqslant$) form but expanded to include the ($\leqslant$), (=), and ($\geqslant$) forms. The third section introduces the proposed approach and numerical examples. The fourth section discusses some important related topics. Finally, the conclusion section summarizes our findings and makes suggestions for future research.

2. Review of existing approaches for shadow price/cost

Consider the LP model in the following general form–standard form in this paper:

$$\begin{aligned} &\max\ Z = \sum_{j=1}^{n} c_j x_j \\ &\text{s.t.}\quad \sum_{j=1}^{n} a_{ij} x_j \begin{Bmatrix} \leqslant \\ = \\ \geqslant \end{Bmatrix} b_i, \quad i = 1,2,\ldots,m, \\ &\qquad x_j \geqslant 0, \quad j = 1,2,\ldots,n. \end{aligned} \tag{2.1}$$

Denote its dual problem as:

$$\begin{aligned} &\min\ Z_D = \sum_{i=1}^{m} b_i y_i \\ &\text{s.t.}\quad \sum_{i=1}^{m} a_{ij} y_i \geqslant c_j, \quad j = 1,2,\ldots,n, \\ &\qquad y_i \begin{Bmatrix} \geqslant 0 \\ \in \Re \\ \leqslant 0 \end{Bmatrix}, \quad i = 1,2,\ldots,m. \end{aligned} \tag{2.2}$$

Furter denote matrices $\mathbf{c}_0 = [c_1, c_2, \ldots, c_n]^t$, $\mathbf{x}_0 = [x_1, x_2, \ldots, x_n]^t$,

$$\mathbf{A}_0 = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad \mathbf{b} = [b_1, b_2, \ldots, b_m]^t \geqslant \mathbf{0}.$$

In common economic interpretation, problem (2.1) is depicted as a resource allocation model in which the objective is to maximize profit. However, the problem is subject to limited resources despite the fact that functional constraint is in the ($\leqslant$), (=) or ($\geqslant$) form. Decision variable x_j represents the level of activity j; coefficient c_j represents the unit profit from activity j. Resource i, whose maximum available amount is b_i, is consumed at the rate a_{ij} units per unit of activity j. This study interprets parameters in the model in the same way.

We roughly classify the approaches for finding true shadow price/cost into two groups. The first group applies the *parametric programming technique* and the second group utilizes the optimal solution set, including the *optimal basis approach*, *convex analysis approach*,*optimal value approach*, *optimal solution approach* and o*optimal partition approach*.

2.1. Parametric programming approach

Let p_i^- and p_i^+ be the negative and positive shadow price of resource i in problem (2.1), respectively. Denote $f(\mathbf{b} + t\mathbf{e}_i)$ as the optimal value function for (2.1) when right-hand-side vector $\mathbf{b}$ is perturbed along the direction $\mathbf{e}_i$, where $\mathbf{e}_i$ is the elementary column vector. Akgül (1984) showed that p_i^- and p_i^+ are the left-sided and right-sided slopes of $f(\mathbf{b} + t\mathbf{e}_i)$ at t=0. Thus

$$p_i^- = \left.\frac{\partial f(\mathbf{b} + t\mathbf{e}_i)}{\partial t^-}\right|_{t=0} \text{ and } p_i^+ = \left.\frac{\partial f(\mathbf{b} + t\mathbf{e}_i)}{\partial t^+}\right|_{t=0}, \quad i = 1,2,\ldots,m \tag{2.3}$$

Hence, p_i^- and p_i^+ can be determined by the traditional parametric programming technique.

This approach can also be used to obtain the shadow price of a resource bundle. Let $\mathbf{v}$ be the resource-bundle vector of $\mathbf{b}$, *i.e.*, bundle of several resources of $\mathbf{b}$, $F(\mathbf{b} + t\mathbf{v})$ be the corresponding optimal value function, and $p_\mathbf{v}^-$ and $p_\mathbf{v}^+$ be the negative and positive shadow price of resource bundle $\mathbf{v}$ in problem (2.1). The $p_\mathbf{v}^-$ and $p_\mathbf{v}^+$ can be determined by

$$p_\mathbf{v}^- = \left.\frac{\partial F(\mathbf{b} + t\mathbf{v})}{\partial t^-}\right|_{t=0} \text{ and } p_\mathbf{v}^+ = \left.\frac{\partial F(\mathbf{b} + t\mathbf{v})}{\partial t^+}\right|_{t=0}. \tag{2.4}$$

Again, they can be determined by the parametric programming technique. In case $\mathbf{v} = \mathbf{e}_i$, formula (2.4) simplifies to (2.3). Thus, formula (2.4) is the general form of formula (2.3). Vector $\mathbf{v}$ is called resource-bundle vector in the rest of the paper.

If $g(\mathbf{c} + t\mathbf{e}_i)$ is the optimal value function for (2.1) when cost coefficient vector $\mathbf{c}$ is perturbed along the direction $\mathbf{e}_i$, the negative and positive shadow costs corresponding to activity j, denoted as q_j^- and q_j^+, respectively, will be the left-sided and right-sided slopes of $g(\mathbf{c} + t\mathbf{e}_i)$ at $t = 0$. Similarly, the negative and positive shadow costs of activity-bundle vector $\mathbf{w}$, denoted as $q_\mathbf{w}^-$ and $q_\mathbf{w}^+$, respectively, are the left-sided and right-sided slopes of optimal value function $G(\mathbf{c} + t\mathbf{w})$ at $t = 0$. Parametric programming technique can also be applied to determine $q_j^-, q_j^+, q_\mathbf{w}^-$ and $q_\mathbf{w}^+$.

Parametric programming technique can correctly obtain the true shadow prices/costs. However degeneracy causes stalling and cycling which make the simplex method and also the parametric programming technique inefficient (Zörnig & Gal, 1996). The larger the degenerate degree of LP, the more inefficient these methods are. Hence, the parametric programming technique is tedious for computing true shadow prices/costs because degeneracy is precisely the reason why two-sided shadow prices/costs have to be defined.

Define $\mathscr{P}^*$ as the set of optimal solutions to (2.1), and $\mathscr{D}^*$ as the set of optimal solutions to (2.2). As for the second group of approaches for finding relative true shadow prices/costs in LP model, Ho (2000) gave the following fundamental and important proposition.

Theorem 1a. *For any LP problem in the standard form,*

$$p_i^- = \max\{y_i : \mathbf{y} \in \mathscr{D}^*\} \quad \textit{and} \quad p_i^+ = \min\{y_i : \mathbf{y} \in \mathscr{D}^*\}. \tag{2.5}$$

There are similar propositions for q_j^- and q_j^+ (Roos, Terlaky, & Vial, 1997, chap. 19).

Theorem 1b. *For any LP problem,*

$$q_j^- = \min\{x_j : \mathbf{x} \in \mathscr{P}^*\} \quad \textit{and} \quad q_j^+ = \max\{x_j : \mathbf{x} \in \mathscr{P}^*\}. \tag{2.6}$$

Formulas (2.5) and (2.6) can easily be expanded to a general case to determine values of $p_\mathbf{v}^-, p_\mathbf{v}^+$ (Akgül, 1984), $q_\mathbf{w}^-$ and $q_\mathbf{w}^+$.