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A hybrid variable neighborhood search and simulated annealing algorithm to estimate the three parameters of the Weibull distribution

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ABSTRACT

The Weibull distribution plays an important role in failure distribution modeling of reliability research. While there are three parameters in the general form of this distribution, for simplicity, one of its parameters is usually omitted and as a result, the others are estimated easily. However, due to its more flexibility, when the general form of the Weibull distribution is of interest, the estimation procedure is not an easy task anymore. For example, in the maximum likelihood estimation method, the likelihood function that is formed for a three-parameter Weibull distribution is very hard to maximize. In this paper, a new hybrid methodology based on a variable neighborhood search and a simulated annealing approach is proposed to maximize the likelihood function of a three-parameter Weibull distribution. The performance of the proposed methodology in terms of both the estimation accuracy and the required CPU time is then evaluated and compared to the ones of an existing current method through a wide range of numerical examples in which a sensitivity analysis is performed on the sample size. The results of the comparison study show that while the proposed method provides accurate estimates as well as those of the existing method, it requires significantly less CPU time.

1. Introduction

It is well-known that the three-parameter Weibull distribution family is extremely flexible and can fit an extremely wide range of empirical observations very well (Nosal, Legge, & Krupa, 2000). It exhibits a wide range of shapes for the density and hazard functions that are suitable to model complex failure data sets. Moreover, it is especially useful as a failure model in analyzing the reliability of different types of systems.

The Weibull probability density and cumulative probability functions are given in (1) and (2), respectively (Johnson & Kotz, 1970).

$$f(x) = \frac{\beta}{\eta}\left(\frac{x-v}{\eta}\right)^{\beta-1} e^{-\left(\frac{x-v}{\eta}\right)^{\beta}}; \quad \beta > 0, \ \eta > 0, \ v > x \tag{1}$$

$$F(x) = 1 - e^{-\left(\frac{x-v}{\eta}\right)^{\beta}} \tag{2}$$

where β, η, and v are the shape, scale, and location parameters, respectively.

Successful application of Weibull distribution depends on having reliable statistical estimates of the three parameters. Estimating the parameters of the three-parameter Weibull distribution family is essentially a very difficult task and because of this difficulty, despite its inherent flexibility, it is seldom used. Even the estimation process of the popular two-parameter Weibull distribution does not offer close estimates and relies on numerical procedures.

Nosal and Nosal (2003) used Monte Carlo simulation and array processing language to investigate the performance of the gradient random search minimization procedure for fitting a Weibull distribution to a given data set using minimum Kolmogorov–Smirnov distance approach. Abbasi, Eshragh-Jahromi, Arkat, and Hosseinkouchack (2006) focused on likelihood method to estimate the parameters of a three-parameter Weibull distribution and employed a simulated annealing (SA) approach to maximize the likelihood function. Simulated annealing is an algorithm that originates in material science engineering and originally introduced to find the equilibrium configuration of a collection of atoms at a given temperature (Metropolis, Rosenbluth, Rosenbluth, Teller, & Teller, 1958). Later on, Kirkpatrick, Gerlatt, and Vecchi (1983) were the first to use it as an algorithm to solve optimization problems. This algorithm is a very effective and easy to implement method that helps engineers find their way through hard optimization problems.

In order to improve the performance of the SA algorithm of Abbasi et al. (2006) and reach the solution in shorter amount of computer CPU time, in this paper, a hybrid variable neighborhood search and SA approach is proposed to maximize the likelihood

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function formed to estimate the parameters of a three-parameter Weibull distribution.

The rest of the paper is organized as follows. In the next section the notion of the maximum likelihood estimation (MLE) is briefly introduced. Sections 3 and 4 give some brief information on the variable neighborhood search (VNS) and SA algorithms. The proposed hybrid methodology is first introduced in Section 5 and then its performance is improved in Section 6. Three different problems are used to investigate the performance of the proposed methodology and to compare it with the one of Abbasi et al. (2006) method in Section 7. Section 8 concludes the current work followed by introducing some future research to those interested in parameter estimation of a three-parameter Weibull distribution using meta-heuristics.

2. Parameter estimation using MLE

Estimation theory is a cornerstone in statistical analyses and several techniques have been introduced to estimate parameters, of which MLE, graphical procedure (Ross, 1994), moments (Lehman, 1962; White, 1969), artificial neural network (Abbasi, Rabelo, & Hosseinkouchack, 2008), and weighted least square method are some of the most interesting ones (Bain & Antle, 1967; Usher, 1996).

For the MLE estimators that are asymptotically unbiased with minimum variances (Dubey, 1965), let $x_1, x_2, \ldots, x_n$ be a random sample of size n drawn, at random, from a probability density function, $f(x;\vec{\theta})$of unknown parameter vector, $\vec{\theta}$. Then the likelihood function is as follows:

$$L = \prod_{i=1}^{n} f_{x_i}(x_i, \vec{\theta}) \tag{3}$$

where $\vec{\theta}$ is a vector of size m representing the unknown parameters, i.e. $\vec{\theta} = (\theta_1, \ldots, \theta_m)$.

The goal is to find a vector, say $\vec{\theta}_0$, that maximizes the so-called likelihood function. To maximize L, we may equivalently use its logarithm, $Ln(L)$. Estimates are hence obtained through solving the following equation set:

$$\frac{\partial}{\partial \vec{\theta}} Ln(L) = \vec{o} \tag{4}$$

For a Weibull distribution, the L function and its logarithm are given in (5) and (6), where $\vec{\theta} = (\beta, \eta, \nu)$.

$$L(x_1, \ldots, x_n; \beta, \eta, \nu) = \prod_{i=1}^{n} \frac{\beta}{\eta}\left(\frac{x_i - \nu}{\eta}\right)^{\beta-1} e^{-\left(\frac{x_i-\nu}{\eta}\right)^{\beta}}; \qquad \beta > 0,\ \ \eta > 0,\ \ x > \nu \tag{5}$$

$$Ln(L(x_1, \ldots, x_n, \beta, \eta, \nu) = nLn\left(\frac{\beta}{\eta}\right) + \sum_{i=1}^{n}\left(+\left(\frac{x_i - \nu}{\eta}\right)^{\beta} + (\beta - 1)Ln\left(\frac{x_i - \nu}{\eta}\right)\right) \tag{6}$$

It is very difficult to maximize L(or $Ln(L)$) in (5) (or (6)) using ordinary optimization techniques (Thoman, Bain, & Antle, 1969). The gradient method being dependent on the partial derivations of the objective function is not a good method to use here, because it is very hard to evaluate the gradient terms and the objective function itself at different points the algorithm needs. Also worth noting, it is quite boring to derive the gradient of this complicated objective function. Hence, some meta-heuristic search techniques need to be employed.

3. General variable neighborhood search (VNS) algorithm

The basic idea of VNS is to explore different neighborhoods for the solution space whenever a local optimum is reached by using a local search method. The VNS is rapidly becoming a well-established method in meta-heuristics (see for instance (Hansen, Mladenovic, & Pérez-Brito, 2001)). VNS is based on a simple and effective idea: a systematic change of neighborhood within a local search algorithm (Garcia, Dionisio, Campos, & Marti, 2006).

The VNS algorithm works in two phases. In the first phase, a set of neighborhood structures N_k, $k = 1, \ldots, k_{max}$, in which N_k is the k^{th} neighborhood, is first defined. Then, an initial solution x is found and a stopping criterion is determined. In the second phase, given the initial solution x, a random point x' in $N_k(x)$ is generated. Starting from x', a local search is then performed to generate x''. If x'' is better than the incumbent best solution x, then $x = x''$, and the search returns to N_1. Otherwise, the search explores the next neighborhood N_{k+1}. This is repeated until $k = k_{max}$.

In other words, after defining the neighborhood structure and a stopping criterion, the phases involved in a VNS method are as follows:

Phase 1: Generating initial solution Set $k = 1$.

Step 1. Find an initial solution x.
Step 2. Generate a random point x' in $N_k(x)$ neighborhood of the current point. This step is called *shaking*.
Step 3. Find a local optimum from the point generated in the previous step. This step is called *local search*.
Step 4. If the new local optimum is better than the previous one, it becomes the current solution; restart the process. Otherwise increase k by one and go to step 1. This step is called *move or not* (Duran & Toksarı, 2007).

Phase 2: Continuous VNS

The pseudo-code of the general continuous VNS is outlined in Algorithm 1.

Algorithm 1. Pseudo-code of the continuous VNS algorithm

```
Select the set of neighborhood structures, N_k, k = 1,...,k_max
   and the array of random distributions types;
Choose an arbitrary initial point x ∈ S
Set x → x* and f(x) → f*
Until the stopping condition is reached, repeat the following
   steps;
  Set k → 1
  Until k_max < k repeat the following steps
    Shaking: Generate at random a point y ∈ N_k(x*).
    Use the local search method to find a local optimum y'.
    Move or not: If f(y') < f* (depending on the objective
  function)
    then set x → y', f(y') → f* and r → 1
    Else
      Update r
    End if
    Set k + 1 → k
  End
End
When stopping condition is met, the point x* is an
   approximate solution of the problem
```

A variety of useful applications of VNS can be found in Melian & Mladenovic (2007).

The neighborhood of the VNS method is structured by $y' \to y + d\lambda r$ where d is the direction of the new neighborhood and takes either 1 or -1, λ is random number and $r$is the radius of neighborhood generator of the VNS algorithm. Furthermore, in