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# A mathematical programming approach to multi-attribute decision making with interval-valued intuitionistic fuzzy assessment information

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ABSTRACT

This article proposes an approach to handle multi-attribute decision making (MADM) problems under the interval-valued intuitionistic fuzzy environment, in which both assessments of alternatives on attributes (hereafter, referred to as attribute values) and attribute weights are provided as interval-valued intuitionistic fuzzy numbers (IVIFNs). The notion of relative closeness is extended to interval values to accommodate IVIFN decision data, and fractional programming models are developed based on the Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) method to determine a relative closeness interval where attribute weights are independently determined for each alternative. By employing a series of optimization models, a quadratic program is established for obtaining a unified attribute weight vector, whereby the individual IVIFN attribute values are aggregated into relative closeness intervals to the ideal solution for final ranking. An illustrative supplier selection problem is employed to demonstrate how to apply the proposed procedure.



## 1. Introduction

Multi-attribute decision making (MADM) handles decision situations where a set of alternatives (usually discrete) has to be assessed against multiple attributes or criteria before a final choice is selected (Hwang & Yoon, 1981). MADM problems may arise from decisions in our daily life as well as complicated decisions in a host of fields such as economics, management and engineering. For instance, when deciding which car to buy, a customer may consider a number of cars by assessing their prices, security, driving experience, quality, and color. It is understandable that the aforesaid five attributes in this decision problem are likely to play different roles in reaching a final purchase decision. These varying roles are typically reflected as different attribute weights in MADM. Eventually, the customer has to aggregate his/her individual assessments of different cars against each attribute into an overall evaluation and selects a car that yields the best overall value. This simple example reveals the three key components in a multi-attribute decision model: attribute values or performance measures, attribute weights, and a mechanism to aggregate this information into an aggregated value or assessment for each alternative.

Due to ambiguity and incomplete information in many decision problems, it is often difficult for a decision-maker (DM) to give his/her assessments on attribute values and weights in crisp values. Instead, it has become increasingly common that these assessments are provided as fuzzy numbers (FNs) or intuitionistic fuzzy numbers (IFNs), leading to a rapidly expanding body of literature on MADM under the fuzzy or intuitionistic fuzzy framework (Atanassov, Pasi, & Yager, 2005; Boran, Genc, Kurt, & Akay, 2009; Hong & Choi, 2000; Li, 2005; Li, Wang, Liu, & Shan, 2009; Liu & Wang, 2007; Szmidt & Kacprzyk, 2002, 2003; Tan & Chen, 2010; Wang, Li, & Wang, 2009; Wang & Qian, 2007; Xu, 2007a, 2007b; Xu & Yager, 2008; Zhang, Zhang, Lai, & Lu, 2009). The notion of intuitionistic fuzzy sets (IFSs) is proposed by Atanassov (1986) to generalize the concept of fuzzy sets. In a fuzzy set, the membership of an element to a particular set is defined as a continuous value between 0 and 1, thereby extending the traditional 0–1 crisp logic to fuzzy logic (Karray & de Silva, 2004). IFSs move one step further by considering not only the membership but also the nonmembership of an element to a given set.

In an IFS, the membership and nonmembership functions are defined as real values between 0 and 1. By allowing these real-valued membership and nonmembership functions to assume interval values, Atanassov and Gargov (1989) extend the notion of IFSs to interval-valued intuitionistic fuzzy sets (IVIFSs). In recent years, the academic community has witnessed growing research interests in IVIFSs, such as investigations on basic operations and relations of IVIFSs as well as their basic properties (Atanassov, 1994; Bustince & Burillo, 1995; Hong, 1998; Hung & Wu, 2002; Xu & Chen, 2008), topological properties (Mondal & Samanta, 2001), relationships

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between IFSs, $L$-fuzzy sets, interval-valued fuzzy sets and IVIFSs (Deschrijver, 2007, 2008; Deschrijver & Kerre, 2007), the entropy and subsethood (Liu, Zheng, & Xiong, 2005), and distance measures and similarity measures of IVIFSs (Xu & Chen, 2008). With this enhanced understanding of IVIFNs, researchers have turned their attention to decision problems where some raw decision data are provided as IVIFNs (Xu, 2007b; Xu & Yager, 2008; Wang et al., 2009). In the existing research on MADM with IVIFN assessments, it is generally assumed that attribute values are given as IVIFNs, but attribute weights are either provided as crisp values or expressed as a set of linear constraints (Wang et al., 2009). In this research, the focus is to consider MADM situations where both attribute values and weights are furnished as IVIFNs.

The remainder of this paper is organized as follows. Section 2 provides some preliminary background on IFSs and IVIFSs. In Section 3, fractional programs and quadratic programs are derived from TOPSIS and a corresponding approach is designed to solve MADM problems with interval-valued intuitionistic fuzzy assessments. Section 4 presents a numerical example to demonstrate how to apply the proposed approach, followed by some concluding remarks in Section 5.

## 2. Preliminaries

This section reviews some basic concepts on IFSs and IVIFSs to make the article self-contained and facilitate the discussion of the proposed method.

**Definition 2.1** (*Atanassov, 1986*). Let $Z$ be a fixed nonempty universe set, an intuitionistic fuzzy set (IFS) $A$ in $Z$ is defined as

$$A = \{\langle z, \mu_A(z), \nu_A(z)\rangle | z \in Z\},$$

where $\mu_A : Z \to [0,1]$ and $\nu_A : Z \to [0,1]$, satisfying $0 \leqslant \mu_A(z) + \nu_A(z) \leqslant 1$, $\forall z \in Z$.

$\mu_A(z)$ and $\nu_A(z)$ are called, respectively, the membership and nonmembership functions of IFS $A$. In addition, for each IFS $A$ in $Z$, $\pi_A(z) = 1 - \mu_A(z) - \nu_A(z)$ is often referred to as its intuitionistic fuzzy index, representing the degree of indeterminacy or hesitation of $z$ to $A$. It is obvious that $0 \leqslant \pi_A(z) \leqslant 1$ for every $z \in Z$.

When the range of the membership and nonmembership functions of an IFS is extended to interval values rather than exact numbers, IFSs become interval-valued intuitionistic fuzzy sets (IVIFSs) (Atanassov & Gargov, 1989).

**Definition 2.2** (*Atanassov and Gargov, 1989*). Let $Z$ be a non-empty set of the universe, and $D[0,1]$ be the set of all closed subintervals of $[0,1]$, an interval-valued intuitionistic fuzzy set (IVIFS) $\widetilde{A}$ over $Z$ is an object in the following form:

$$\widetilde{A} = \{\langle z, \tilde{\mu}_{\widetilde{A}}(z), \tilde{\nu}_{\widetilde{A}}(z)\rangle | z \in Z\},$$

where $\tilde{\mu}_{\widetilde{A}} : Z \to D[0,1]$, $\tilde{\nu}_{\widetilde{A}} : Z \to D[0,1]$, and $0 \leqslant \sup(\tilde{\mu}_{\widetilde{A}}(z)) + \sup(\tilde{\nu}_{\widetilde{A}}(z)) \leqslant 1$ for any $z \in Z$.

The intervals $\tilde{\mu}_{\widetilde{A}}(z)$ and $\tilde{\nu}_{\widetilde{A}}(z)$ denote, respectively, the degree of membership and nonmembership of $z$ to $A$. For each $z \in Z$, $\tilde{\mu}_{\widetilde{A}}(z)$ and $\tilde{\nu}_{\widetilde{A}}(z)$ are closed intervals and their lower and upper boundaries are denoted by $\tilde{\mu}^L_{\widetilde{A}}(z)$, $\tilde{\mu}^U_{\widetilde{A}}(z)$, $\tilde{\nu}^L_{\widetilde{A}}(z)$ and $\tilde{\nu}^U_{\widetilde{A}}(z)$. Therefore, another equivalent way to express IVIFS $\widetilde{A}$ is

$$\widetilde{A} = \{\langle z, [\tilde{\mu}^L_{\widetilde{A}}(z), \tilde{\mu}^U_{\widetilde{A}}(z)], [\tilde{\nu}^L_{\widetilde{A}}(z), \tilde{\nu}^U_{\widetilde{A}}(z)]\rangle | z \in Z\},$$

where

$$\tilde{\mu}^U_{\widetilde{A}}(z) + \tilde{\nu}^U_{\widetilde{A}}(z) \leqslant 1,\ 0 \leqslant \tilde{\mu}^L_{\widetilde{A}}(z) \leqslant \tilde{\mu}^U_{\widetilde{A}}(z) \leqslant 1,\ 0 \leqslant \tilde{\nu}^L_{\widetilde{A}}(z) \leqslant \tilde{\nu}^U_{\widetilde{A}}(z) \leqslant 1.$$

Similar to IFSs, for each element $z \in Z$, its hesitation interval relative to $\widetilde{A}$ is given as:

$$\tilde{\pi}_{\widetilde{A}}(z) = [\tilde{\pi}^L_{\widetilde{A}}(z), \tilde{\pi}^U_{\widetilde{A}}(z)] = [1 - \tilde{\mu}^U_{\widetilde{A}}(z) - \tilde{\nu}^U_{\widetilde{A}}(z), 1 - \tilde{\mu}^L_{\widetilde{A}}(z) - \tilde{\nu}^L_{\widetilde{A}}(z)].$$

Especially, for every $z \in Z$, if

$$\mu_{\widetilde{A}}(z) = \tilde{\mu}^L_{\widetilde{A}}(z) = \tilde{\mu}^U_{\widetilde{A}}(z), \quad \nu_{\widetilde{A}}(z) = \tilde{\nu}^L_{\widetilde{A}}(z) = \tilde{\nu}^U_{\widetilde{A}}(z)$$

then, IVIFS $\widetilde{A}$ reduces to an ordinary IFS.

For an IVIFS $\widetilde{A}$ and a given $z$, the pair $(\tilde{\mu}_{\widetilde{A}}(z), \tilde{\nu}_{\widetilde{A}}(z))$ is called an interval-valued intuitionistic fuzzy number (IVIFN) (Wang et al., 2009; Xu & Yager, 2008). For convenience, the pair $(\tilde{\mu}_{\widetilde{A}}(z), \tilde{\nu}_{\widetilde{A}}(z))$ is often denoted by $([a,b],[c,d])$, where $[a,b] \in D[0,1]$, $[c,d] \in D[0,1]$ and $b + d \leqslant 1$.

After the initial decision data in IVIFNs are processed, the proposed model will generate an aggregated relative closeness interval, expressed as an IVIFN, to the ideal solution for each alternative. To make a final choice based on the aggregated relative closeness intervals, it is necessary to examine how to rank IVIFNs. Xu (2007b) introduces the score and accuracy functions for IVIFNs and applies them to compare two IVIFNs. Wang et al. (2009) note that many distinct IVIFNs cannot be differentiated by these two functions. As such, two new functions, the membership uncertainty index and the hesitation uncertainty index, are defined therein. Along with the score and accuracy functions, Wang et al. (2009) devise a unique prioritized IVIFN comparison approach that is able to distinguish any two distinct IVIFNs. This same comparison approach will be adopted in this research for ranking alternatives based on IVIFNs. Next, these four functions are defined.

**Definition 2.3** (*Xu, 2007b*). For an IVIFN $\tilde{\alpha} = ([a,b],[c,d])$, its score function is defined as $S(\tilde{\alpha}) = \frac{a+b-c-d}{2}$.

**Definition 2.4** (*Xu, 2007b*). For an IVIFN $\tilde{\alpha} = ([a,b],[c,d])$, its accuracy function is defined as $H(\tilde{\alpha}) = \frac{a+b+c+d}{2}$.

**Definition 2.5** (*Wang et al., 2009*). For an IVIFN $\tilde{\alpha} = ([a,b],[c,d])$, its membership uncertainty index is defined as $T(\tilde{\alpha}) = b + c - a - d$.

**Definition 2.6** (*Wang et al., 2009*). For an IVIFN $\tilde{\alpha} = ([a,b],[c,d])$, its hesitation uncertainty index is defined as $G(\tilde{\alpha}) = b + d - a - c$.

For a discussion of these four functions and their properties, readers are referred to Wang et al. (2009). Based on these functions, a prioritized comparison method is introduced as follows.

**Definition 2.7** (*Wang et al., 2009*). For any two IVIFNs $\tilde{\alpha} = ([a_1,b_1],[c_1,d_1])$ and $\tilde{\beta} = ([a_2,b_2],[c_2,d_2])$,

If $S(\tilde{\alpha}) < S(\tilde{\beta})$, then $\tilde{\alpha}$ is smaller than $\tilde{\beta}$, denoted by $\tilde{\alpha} < \tilde{\beta}$;
If $S(\tilde{\alpha}) > S(\tilde{\beta})$, then $\tilde{\alpha}$ is greater than $\tilde{\beta}$, denoted by $\tilde{\alpha} > \tilde{\beta}$;
If $S(\tilde{\alpha}) = S(\tilde{\beta})$, then
(1) If $H(\tilde{\alpha}) < H(\tilde{\beta})$, then $\tilde{\alpha}$ is smaller than $\tilde{\beta}$, denoted by $\tilde{\alpha} < \tilde{\beta}$;
(2) If $H(\tilde{\alpha}) > H(\tilde{\beta})$, then $\tilde{\alpha}$ is greater than $\tilde{\beta}$, denoted by $\tilde{\alpha} > \tilde{\beta}$;
(3) If $H(\tilde{\alpha}) = H(\tilde{\beta})$, then
 (i) If $T(\tilde{\alpha}) > T(\tilde{\beta})$, then $\tilde{\alpha}$ is smaller than $\tilde{\beta}$, denoted by $\tilde{\alpha} < \tilde{\beta}$;
 (ii) If $T(\tilde{\alpha}) < T(\tilde{\beta})$, then $\tilde{\alpha}$ is greater than $\tilde{\beta}$, denoted by $\tilde{\alpha} > \tilde{\beta}$;
 (ii) If $T(\tilde{\alpha}) = T(\tilde{\beta})$, then
 (a) If $G(\tilde{\alpha}) > G(\tilde{\beta})$, then $\tilde{\alpha}$ is smaller than $\tilde{\beta}$, denoted by $\tilde{\alpha} < \tilde{\beta}$;
 (b) If $G(\tilde{\alpha}) < G(\tilde{\beta})$, then $\tilde{\alpha}$ is greater than $\tilde{\beta}$, denoted by $\tilde{\alpha} > \tilde{\beta}$;
 (c) If $G(\tilde{\alpha}) = G(\tilde{\beta})$, then $\tilde{\alpha}$ and $\tilde{\beta}$ represent the same information, denoted by $\tilde{\alpha} = \tilde{\beta}$.

For any two IVIFNs, $\tilde{\alpha}$ and $\tilde{\beta}$, denote $\tilde{\alpha} \leqslant \tilde{\beta}$ iff $\tilde{\alpha} < \tilde{\beta}$ or $\tilde{\alpha} = \tilde{\beta}$.