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Fuzzy Sets and Systems 300 (2016) 24–39

FUZZY sets and systems

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Fuzzifying topology induced by a strong fuzzy metric

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Received 12 February 2015; received in revised form 1 November 2015; accepted 5 November 2015

Available online 10 November 2015

Abstract

A construction of a fuzzifying topology induced by a strong fuzzy metric is presented. Properties of this fuzzifying topology, in particular, its convergence structure are studied. Our special interest is in the study of the relations between products of fuzzy metrics and the products of the induced fuzzifying topologies.

Keywords: Fuzzy (pseudo-)metric; Fuzzifying topology; Continuous mapping of fuzzy (pseudo-)metric spaces; Continuous mapping of fuzzifying topological spaces

1. Introduction

After fuzzy metric was defined by I. Kramosil and J. Michalek [13] and later redefined in a slightly different way by A. George and P. Veeramani [4], many researchers became interested in the topological structure of a fuzzy metric space. In particular, properties of topologies induced by fuzzy metrics were studied by A. George and P. Veeramani, V. Gregori, S. Romaguera, A. Sapena, D. Mihet, S. Morilas et al., see e.g. [4,5,10,9,6,7,18]. In most papers, the topology induced by a fuzzy metric is actually an *ordinary*, that is a *crisp* topology on the underlying set. However recently some authors showed interest in a *fuzzy-type topological structures* induced by fuzzy (pseudo-)metrics, see [30,15].[1] It is also the principal goal of the present paper to study this problem.

To state our idea more precisely, we recall the three basic approaches to the concept of a fuzzy topology.[2] The first approach was initiated by Zadeh's student C.L. Chang [2] and soon it was conceptually generalized by J.A. Goguen [3]. It realizes an L-fuzzy topology on a set X (where L is a complete lattice, or more generally a cl-monoid) as a certain crisp subset T of the L-powerset of a set X, that is $T \subseteq L^X$. We refer to this view on a fuzzy topology as *a crisp-fuzzy approach*. The second approach, first presented by U. Höhle [11] and then independently

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[1] We are grateful to an anonymous referee for paying our attention to these works.

[2] Here we restrict ourselves to the fixed basis fuzzy topologies as it is specified by S.E. Rodabaugh [19].

http://dx.doi.org/10.1016/j.fss.2015.11.005
0165-0114/

rediscovered by M.S. Ying [26], realizes an L-fuzzy topology $\mathcal{T}$ on a set X as an L-fuzzy subset of the powerset of X, that is as a mapping $\mathcal{T}: 2^X \to L$, satisfying certain conditions. Following M.S. Ying, such structures are usually called fuzzifying topologies. We view this approach as *a fuzzy-crisp* one. Finally, the last one of the three approaches interprets a fuzzy topology as a fuzzy subset $\mathcal{T}$ of the fuzzy powerset of a set X, that is as a mapping $\mathcal{T}: L^X \to L$. It was first presented (independently) by T. Kubiak [14] and by the second named author of this paper [23,24]. We call it by *a fuzzy-fuzzy* approach.

Developing the general fuzzy viewpoint on the topological structure of a fuzzy metric space, here we (as well as the authors of the both above mentioned papers [30,15]) start with the fuzzy-crisp approach. It is just the main goal of the present paper to develop the foundations of this approach, that is to work out the concept of a fuzzifying topology induced by a strong fuzzy metric and to present our results obtained in this field so far.

The structure of the paper is as follows. The next section, Preliminaries, is divided into four, rather independent subsections, in which we expose basic definitions and results concerning fuzzy metrics that are needed in the sequel, give an introduction into the theory of fuzzifying topologies, present a construction of a fuzzifying topology from an ordered family of topologies, and apply this construction to describe the product of fuzzifying topologies. In the third section, we describe the construction of a fuzzifying topology induced by a strong fuzzy metric, consider some properties of this construction and describe the convergence structure of a fuzzifying topology induced by a strong fuzzy metric. In Section 4 we define and study products, separately finite and countable, of fuzzy metrics and show that the fuzzifying topology induced by products of strong fuzzy metrics coincides with the products of the fuzzifying topologies induced by these fuzzy metrics. In the last Section 5, Conclusions, we state some problems which we could not solve in the process of writing this paper and discuss several prospectives for the future work. Besides we make here some remarks concerning the relations of our approach and the approaches proposed in [30] and [15] to the study of fuzzifying topology induced by a fuzzy metric.

2. Preliminaries

2.1. Fuzzy metrics

Basing on the concept of a statistical metric introduced by K. Menger [17], see also [22], I. Kramosil and J. Michalek [13] introduced the notion of a fuzzy metric. Later A. George and P. Veeramani [4] slightly modified the original concept of a fuzzy metric. At present in most cases research involving fuzzy metrics is done in the context of George–Veeramani definition. This approach is accepted also in our paper.

Let X be a non-empty set, $*: [0,1] \times [0,1] \to [0,1]$ be a continuous t-norm (see e.g. [17,22]) and $\mathbb{R}^+ = (0, +\infty)$.

Definition 2.1. (See [4].) A fuzzy metric on the set X is a pair $(M, *)$, or simply M where $M: X \times X \times \mathbb{R}^+ \to [0,1]$ (that is M is a fuzzy subset of $X \times X \times \mathbb{R}^+$), satisfying the following conditions for all $x, y, z \in X$, $s, t \in \mathbb{R}^+$:

(1GV) $M(x, y, t) > 0$;
(2GV) $M(x, y, t) = 1$ if and only if $x = y$;
(3GV) $M(x, y, t) = M(y, x, t)$;
(4GV) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$;
(5GV) $M(x, y, -): \mathbb{R}^+ \to [0,1]$ is continuous.

If $(M, *)$ is a fuzzy metric on X, then the triple $(X, M, *)$ is called *a fuzzy metric space*.

If axiom (2GV) is replaced by a weaker axiom

(2′GV) if $x = y$, then $M(x, y, t) = 1$

we get definitions of a fuzzy pseudo-metric, and the corresponding fuzzy pseudo-metric space.

Note that axiom (4GV) combined with axiom (2′GV) implies that the fuzzy metric $M(x, y, t)$ is non-decreasing in the third argument.