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Fuzzy fractals and hyperfractals

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Abstract

In analogy of the relationship between crisp multivalued fractals and the associated hyperfractals studied recently by ourselves, the properties of fuzzy fractals are investigated by means of fuzzy hyperfractals. In particular, we focus on their address structure, Hausdorff dimension and visualization. Two illustrative examples are supplied.

Keywords: Fuzzy fractal; Fuzzy hyperfractal; Visualization; Hausdorff dimension

1. Introduction

The seminal idea of fuzzification due to Zadeh [1] found naturally its application also to fractals (see e.g. [2–9], and the references therein). On each level set, a fuzzy fractal is usually an ordinary (crisp) fractal whose Hausdorff dimension can be separately calculated or at least estimated (see e.g. [10,11]). Thus, the Hausdorff dimension of a fuzzy fractal can be expressed as a function of its α-levels (see e.g. [12,8], and the references therein). Nevertheless, only ordinary (single-valued) fractals on all α-levels can be effectively treated in this way.

Our approach in the present paper is different. Since multivalued fractals are allowed to occur on any level set (thus, we should rather speak about fuzzy multivalued fractals with this respect), we benefit from the associated hyperfractals, studied recently by ourselves in [13–15]. More concretely, we associate here analogously to fuzzy fractals the fuzzy hyperfractals whose Hausdorff dimension can be calculated as a whole. The underlying fuzzy fractals can be regarded as "shadows" of the associated fuzzy hyperfractals. Therefore, we can describe their complexity in this manner.

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Besides other things, we show that fuzzy fractals and the associated fuzzy hyperfractals have the same address structure. On this basis, we can visualize fuzzy fractals, a structure of fuzzy hyperfractals and the corresponding invariant measures.

Hence, our paper is organized as follows. After Preliminaries which are divided into three parts, we recall the definition of a fuzzy fractal as an attractor of a fuzzy iterated function system (IFS) and introduce a fuzzy hyperfractal as an attractor of a fuzzy hyperIFS. We will show that fuzzy fractals and the associated fuzzy hyperfractals have the same address structure. The Hausdorff dimension of special fuzzy hyperfractals is then calculated by means of the Moran formula. The coincidence of the address structures also helps us to visualize fuzzy fractals, fuzzy hyperfractals and the corresponding invariant measures. Finally, two illustrative examples demonstrate our approach.

2. Some preliminaries about spaces and hyperspaces

Following Hutchinson [11] and Barnsley [16], fractals will be treated here as fixed points of special (usually called the Hutchinson–Barnsley) operators induced in hyperspaces. Hence, let us recall the notion of a hyperspace, at first.

By a *hyperspace* $(H(X), \mathrm{d_H})$ of a metric space (X, d), we understand a certain class $H(X)$ of nonempty subsets of X endowed with the induced Hausdorff metric $\mathrm{d_H}$, i.e. (cf. e.g. [17–20])

$$\mathrm{d_H}(A, B) := \inf\{r > 0 \mid A \subset O_r(B) \text{ and } B \subset O_r(A)\},$$

where

$$O_r(A) := \{x \in X \mid \exists a \in A\colon \mathrm{d}(x, a) < r)\}$$

and $A, B \in H(X)$. An alternative equivalent definition reads

$$\begin{aligned}\mathrm{d_H}(A, B) &:= \max\left\{\sup_{a\in A} \mathrm{d}(a, B), \sup_{b\in B} \mathrm{d}(b, A)\right\}\\ &= \max\left\{\sup_{a\in A}(\inf_{b\in B} \mathrm{d}(a, b)), \sup_{b\in B}(\inf_{a\in A} \mathrm{d}(a, b))\right\},\end{aligned}$$

where $A, B \in H(X)$.

The following two classes will be under our consideration:

$$\begin{aligned}&K(X) := \{A \subset X \mid A \text{ is nonempty and compact}\},\\ &K_{Co}(X) := \{A \subset X \mid A \text{ is nonempty, compact and convex}\},\end{aligned}$$

of course, provided X is a linear space in the latter case, e.g. $\mathbb{R}^m$.

It is well known (see e.g. [21,19,22,23]) that if (X, d) is a complete (resp. compact) metric space, then so is the hyperspace $(K(X), \mathrm{d_H})$. If (X, d) is a compact convex subset of a Banach space, then $(K_{Co}(X), \mathrm{d_H})$ is, according to [24,25], a compact convex subset of $(K(X), \mathrm{d_H})$.

Convex sets and support functions play the key role in our calculations and visualizations. Let us therefore also recall at least some basic notions and properties related to convex sets, convex hulls and support functions.

For $A \subset \mathbb{R}^m$, $B \subset \mathbb{R}^m$, let us define, as usually,

$$\begin{aligned}&A + B := \{x \mid x = a + b,\ a \in A,\ b \in B\},\\ &c \cdot A := \{x \mid x = c \cdot a,\ a \in A\},\ c \in \mathbb{R}.\end{aligned}$$

Thus, if A and B are convex subsets of $\mathbb{R}^m$, then $A + B$, and $c \cdot A$ are convex as well (see e.g. [21,26]) and, especially, $\mathcal{Q}A$ is convex, for $A \in K_{Co}(\mathbb{R}^m)$, $\mathcal{Q} \in \mathbb{R}^{m\times m}$ (see e.g. [21, Theorem 1.4.1]).

Defining the *convex hull* $\mathrm{conv}(A)$ of $A \in K(\mathbb{R}^m)$ as (see e.g. [26, Chapter V.2])

$$\begin{aligned}\mathrm{conv}(A) := \{x \in \mathbb{R}^m \mid x = &\textstyle\sum_{i=1}^{p} \alpha_i a_i,\ \sum_{i=1}^{n+1} \alpha_i = 1,\\ &\alpha_i \geq 0,\ a_i \in A,\ i = 1, \ldots, p\},\end{aligned}$$

$p = 1, 2, \ldots$, it is obviously the smallest convex set containing $A \subset \mathbb{R}^m$. Without any loss of generality, we can simply fix $p = m + 1$. For any $A, B \subset \mathbb{R}^m$, it holds (see e.g. [26, Lemma V.2.4]):