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The Wijsman topology of a fuzzy metric space

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Abstract

We introduce and study the notions of lower Wijsman topology, upper Wijsman topology and Wijsman topology of a fuzzy metric space in the sense of Kramosil and Michalek. In particular, quasi-uniformizability, uniformizability, quasi-metrizability and metrizability of these topologies are discussed. Their relations with other hypertopologies are also analyzed. Corresponding results to the Wijsman topology of a metric space are deduced from our approach with the help of the standard fuzzy metric.
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1. Introduction and preliminaries

Wijsman introduced in [30] a kind of convergence for sequences of subsets of $\mathbb{R}^n$ which is suitable for working with unbounded sets. Wijsman convergence motivated the introduction and deep study of a topology on the set $\mathcal{C}_0(X)$ of all nonempty closed subsets of a metric space (X, d), the so-called Wijsman topology (see e.g. [1,2] and their references, [10,12,25,31], and [6,5] for more recent contributions). In particular, the Wijsman topology of a metric space (X, d) is weaker than the topology of the Hausdorff distance of (X, d).

Since, on the one hand, Wijsman convergence is considered by many mathematicians as the point of departure for the modern theory of set convergence ([1, Section 2.1, p. 34], [2, Section 1]) and, on the other hand, there exists a well-established theory of the Hausdorff fuzzy metric for fuzzy metric spaces, the problem of extending the notion of Wijsman topology to a fuzzy metric space $(X, M, *)$ and investigate, among other properties, its relation with the topology induced by the Hausdorff fuzzy metric of $(X, M, *)$, arises in a natural way. In Section 2 we explore this problem and show that the situation presents some interesting differences with respect to the classical case of metric spaces. We also establish some fundamental results on uniformizability and metrizability of the Wijsman topology

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of a fuzzy metric space from which we deduce as a consequence the classical results corresponding to metric spaces with the help of the standard fuzzy metric of a metric space.

Furthermore, the Wijsman topology has been considered as the building blocks of the lattice of hypertopologies because several hypertopologies can be obtained by taking supremum or infimum of Wijsman topologies (see [3,11,12]). Then, it is natural to wonder if this also occurs in the fuzzy context. Section 3 will be devoted to start this study.

In the sequel the letters $\mathbb{R}$ and $\mathbb{N}$ will denote the set of real numbers and the set of positive integer numbers, respectively.

Our basic references for quasi-metric spaces and quasi-uniform spaces are [9] and [15], and for general topology it is [14].

Let us recall that a quasi-uniformity on a set X is a filter $\mathcal{U}$ on $X \times X$ such that:

(QU1) for each $U \in \mathcal{U}$, $\Delta \subseteq U$, where $\Delta = \{(x,x) : x \in X\}$;
(QU2) for each $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that $V^2 \subseteq U$, where $V^2 = \{(x,y) \in X \times X$: there is $z \in X$ with $(x,z) \in V$ and $(z,y) \in V\}$.

If, in addition, $\mathcal{U}$ satisfies:

(QU3) for each $U \in \mathcal{U}$, $U^{-1} \in \mathcal{U}$, where $U^{-1} = \{(x,y) \in X \times X : (y,x) \in U\}$,

then $\mathcal{U}$ is called a uniformity (on X).

By a (quasi-)uniform space we mean a pair $(X, \mathcal{U})$ such that X is a set and $\mathcal{U}$ is a (quasi-)uniformity on X.

Given a quasi-uniformity $\mathcal{U}$ on a set X, the filter $\mathcal{U}^{-1}$ defined on $X \times X$ by $\mathcal{U}^{-1} = \{U^{-1} : U \in \mathcal{U}\}$ is also a quasi-uniformity on X, called the conjugate of $\mathcal{U}$, and the filter $\mathcal{U}^s = \mathcal{U} \vee \mathcal{U}^{-1}$ is a uniformity on X. Obviously each uniformity $\mathcal{U}$ is a quasi-uniformity where $\mathcal{U} = \mathcal{U}^{-1}$.

Each quasi-uniformity $\mathcal{U}$ on X induces a topology $\tau_{\mathcal{U}}$ on X such that a neighborhood base for each point $x \in X$ is given by $\{U(x) : U \in \mathcal{U}\}$, where $U(x) = \{y \in X : (x,y) \in \mathcal{U}\}$.

A bitopological space (or simply, a bispace) is a triple (X, τ_1, τ_2) such that X is a set, and τ_1 and τ_2 are topologies on X.

A bispace (X, τ_1, τ_2) is called quasi-uniformizable if there is a quasi-uniformity $\mathcal{U}$ on X such that $\tau_{\mathcal{U}} = \tau_1$ and $\tau_{\mathcal{U}^{-1}} = \tau_2$. If τ_1 is a T_0-topology, we say that (X, τ_1, τ_2) is a quasi-uniformizable T_0-bispace. In this case τ_2 is also a T_0 topology and $(X, \tau_1 \vee \tau_2)$ is a Hausdorff uniformizable topological space.

A quasi-metric on a set X is a function $d : X \times X \to [0, +\infty)$ such that for all $x, y, z \in X$:

(QM1) $d(x,y) = d(y,x) = 0 \Leftrightarrow x = y$;
(QM2) $d(x,y) \leq d(x,z) + d(z,y)$.

We shall also consider extended quasi-metrics. These satisfy the preceding conditions (QM1) and (QM2) above, except that we allow $d(x,y) = +\infty$.

By a quasi-metric space we mean a pair (X, d) such that X is a set and d is a quasi-metric or an extended quasi-metric on X.

Given an extended quasi-metric d on a set X, the function d^{-1} defined on $X \times X$ by $d^{-1}(x,y) = d(y,x)$, is also an extended quasi-metric on X, called the conjugate of d, and the function d^s defined on $X \times X$ by $d^s(x,y) = \max\{d(x,y), d^{-1}(x,y)\}$ is an extended metric on X. Of course, d^{-1} and d^s are a quasi-metric and a metric, respectively, whenever d is a quasi-metric on X. Obviously, each (extended) metric d is an (extended) quasi-metric where $d = d^{-1}$.

The following is an easy but useful example of a quasi-metric space.

Example 1.1. Let d be the function defined on $\mathbb{R} \times \mathbb{R}$ by $d(x,y) = \max\{x - y, 0\}$. Then $(\mathbb{R}, d)$ is a quasi-metric space and d^s is the Euclidean metric on $\mathbb{R}$.

Each extended quasi-metric d on X induces a T_0 topology τ_d on X which has as a base the family of open balls $\{B_d(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_d(x,\varepsilon) = \{y \in X : d(x,y) < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Furthermore, it generates a quasi-uniformity $\mathcal{U}_d$ on X which has as a base the countable family $\{U_n^d : n \in \mathbb{N}\}$, where $U_n^d = \{(x,y) \in X \times X : d(x,y) < 2^{-n}\}$ for all $n \in \mathbb{N}$.

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