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On some classes of nonlinear contractions in probabilistic metric spaces

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Abstract

Motivated by a question in J.-X. Fang (2015) [6], we investigate the existence of fixed points for several classes of probabilistic φ-contractions under Fang-type conditions.

Keywords: Fixed point; Probabilistic φ-contraction; Menger probabilistic metric space

1. Introduction

According to Rus [18], a comparison function is a mapping $\varphi : [0, \infty) \to [0, \infty)$ such that φ is monotone increasing and $(\varphi^n(t))_n$ converges to 0 for all $t \geq 0$. A selfmapping f of a metric space (X, d) is called a φ-contraction if φ is a comparison function and

$$d(f(x), f(y)) \leq \varphi(d(x, y)),\ \forall x, y \in X.$$

A well-known theorem of Matkowski [12] states that every φ-contraction on a complete metric space is a Picard mapping, i.e., it has a unique fixed point x_* and $(f^n(x_0))_n$ converges to x_*, for any $x_0 \in X$. It is also known (see, e.g., [11,13]) that the monotonicity of φ in the above theorem cannot be dropped.

Following the deterministic case, as a natural generalization of Sehgal contractions, the class of probabilistic φ-contractions is defined by means of a function φ with suitable properties.

Definition 1.1. A probabilistic φ-contraction on a Menger PM space (X, F, Δ) is a mapping $T : X \to X$ satisfying

$$F_{Tx,Ty}(\varphi(t)) \geq F_{x,y}(t),\ \forall x, y \in X, t > 0,$$

for some given function $\varphi : [0, \infty) \to [0, \infty)$.

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Many papers (e.g. [4,5,9,10,19]) have identified various conditions on the mapping φ which guarantee the existence and uniqueness of the fixed points for probabilistic φ-contractions on a complete Menger PM space. Generally (see [4] and the references in [2]) the fixed point theorems for probabilistic φ-contractions were obtained under the assumption that φ is nondecreasing and $\sum_{n=1}^{\infty} \varphi^n(t) < \infty$ for any $t > 0$. These conditions were significantly weakened in the recent papers [2,11,6].

Theorem 1.1. *(See [2,11].) Let (X, F, Δ) be a complete Menger PM space such that Δ is a continuous triangular norm of H-type. Let a function $\varphi : [0, \infty) \to [0, \infty)$ be such that, for any $t > 0$,*

$$0 < \varphi(t) < t \text{ and } \lim_{n\to\infty} \varphi^n(t) = 0.$$

If $T : X \to X$ is a probabilistic φ-contraction, then T has a unique fixed point x_* and for any $x_0 \in X$, $\lim_{n\to\infty} T^n x_0 = x_*$.

In [6] J.-X. Fang showed that the function φ only needs to satisfy the condition:

$$\text{for each } t > 0 \text{ there exists } r \geq t \text{ such that } \lim_{n\to\infty} \varphi^n(r) = 0. \quad (F)$$

The class of functions $\varphi : [0, \infty) \to [0, \infty)$ satisfying (F) will be denoted by Φ_w. By [6, Lemma 3.1], every $\varphi \in \Phi_w$ has the following property:

$$\text{for each } t > 0 \text{ there exists } r \geq t \text{ such that } \varphi(r) < t.$$

Obviously, any function $\varphi : [0, \infty) \to [0, \infty)$ with $\lim_{n\to\infty} \varphi^n(t) = 0$, $\forall t > 0$ belongs to Φ_w.

Theorem 1.2. *(See [6].) Let (X, F, Δ) be a complete Menger PM space with Δ a t-norm of H-type. If $T : X \to X$ is a probabilistic φ-contraction with $\varphi \in \Phi_w$, then T has a unique fixed point.*

Fang asked whether the condition (F) can be used to generalize some other types of fixed point theorems. Motivated by his question, we discuss this problem for three other classes of probabilistic contractions of Sehgal type: probabilistic $(\varphi, \varepsilon - \lambda)$-contractions, probabilistic (φ, b_n)-contractions and probabilistic ψ-contractions.

The terminology and the notations in this paper are those in [6]. For more details related to fixed point theory in probabilistic metric spaces we refer the reader to the books [1] and [8].

We only recall that a t-norm Δ is said to be of H-type [7] if the family of its iterates $\{\Delta^n\}_{n\in\mathbb{N}}$, given by $\Delta^0(x) = 1$, and $\Delta^n(x) = \Delta(\Delta^{n-1}(x), x)$ for all $n \geq 1$, is equicontinuous at $x = 1$. The following theorem provides a characterization of t-norms of H-type.

Theorem 1.3. *(See [16].) (i) Suppose that there exists a strictly increasing sequence $(b_n)_n$ in $[0, 1)$ such that $\lim_{n\to\infty} b_n = 1$ and $\Delta(b_n, b_n) = b_n$. Then Δ is of H-type.*

(ii) Conversely, if Δ is continuous and of H-type, then there exists a sequence $(b_n)_n$ of idempotents of Δ as in (i).

2. Probabilistic $(\varphi, \varepsilon - \lambda)$-contractions

In this section we apply Theorem 1.2 to improve a fixed point result concerning probabilistic $(\varphi, \varepsilon - \lambda)$-contractions.

It is well known that every Sehgal contraction on a complete Menger PM space (X, F, Δ) with Δ continuous has a fixed point iff Δ is of H-type [17,8]. A special class of Sehgal contractions having a fixed point even in complete Menger PM spaces endowed with the Łukasiewicz t-norm is given in the following

Definition 2.1. (See [14].) Let (X, F, Δ) be a Menger PM space. A mapping $T : X \to X$ is called a probabilistic contraction of $(\varepsilon - \lambda)$-type if, for some $k \in (0, 1)$,

$$(\forall \varepsilon > 0, \forall \lambda \in (0, 1)) \; F_{x,y}(\varepsilon) > 1 - \lambda \Rightarrow F_{Tx,Ty}(k\varepsilon) > 1 - k\lambda.$$

More generally one defines the concept of probabilistic $(\varphi, \varepsilon - \lambda)$-contraction.