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On fuzzy ψ-contractive sequences and fixed point theorems

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Abstract

In this paper we give a fixed point theorem in the context of fuzzy metric spaces in the sense of George and Veeramani. As a consequence of our result we obtain a fixed point theorem due to D. Mihet and generalize a fixed point theorem due to D. Wardowski. Also, we answer in a positive way to a question posed by D. Wardowski, and solve partially an open question on Cauchyness and contractivity.

Keywords: Fuzzy metric space; Fuzzy contractive mapping; Fixed point

1. Introduction

In 1975, Kramosil and Michalek [6] gave a notion of fuzzy metric space (*KM*-fuzzy metric space along the paper), which was modified later by George and Veeramani [1] (fuzzy metric space along the paper). Since then, many authors have contributed to the study of these concepts of fuzzy metric. One of the most important topics of research in this field has been the fixed point theory. The first attempt to extend the well-known Banach contraction theorem to *KM*-fuzzy metrics was done by Grabiec in [2]. Later, Gregori and Sapena [5] gave another notion of fuzzy contractive mapping and studied its applicability to fixed point theory in both contexts of fuzzy metrics above mentioned. In their study, the authors needed to demand additional conditions to the completeness of the fuzzy metric in order to obtain a fixed point theorem, which constitutes a significant difference with the classical theory. So, in [5] it was formulated the question ($Q1$): Is a fuzzy contractive sequence a Cauchy sequence (in the sense of George and Veeramani)? D. Mihet showed that the answer to this question in the context of *KM*-fuzzy metric spaces is negative [7, Remark 3.1]. Later, this notion of fuzzy contractive mapping and others that appeared in the literature were generalized by D. Mihet in [7] introducing the concept of fuzzy ψ-contractive mapping and he obtained a fixed point theorem for the class of complete non-Archimedean *KM*-fuzzy metrics.

Recently, D. Wardowski [9] has provided a new contribution to the study of fixed point theory in fuzzy metric spaces. In [9], the author introduced the concept of fuzzy $\mathcal{H}$-contractive mappings (Definition 2.9), which constitutes

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a generalization of the concept given by V. Gregori and A. Sapena, and he obtained the next fixed point theorem for complete fuzzy metric spaces in the sense of George and Veeramani.

Theorem 1.1. *(See Wardowski [9].) Let $(X, M, *)$ be a complete fuzzy metric space and let $f : X \to X$ be a fuzzy $\mathcal{H}$-contractive mapping with respect to $\eta \in \mathcal{H}$ such that:*

(a) $\Pi_{i=1}^{k} M(x, f(x), t_i) \neq 0$, *for* $x \in X$, $k \in \mathbb{N}$ *and any sequence* $\{t_i\} \subset]0, \infty[$, $t_i \searrow 0$*;*
(b) $r * s > 0 \Rightarrow \eta(r * s) \leq \eta(r) + \eta(s)$, *for all* $r, s \in \{M(x, f(x), t) : x \in X, t > 0\}$*;*
(c) $\{\eta(M(x, f(x), t_i)) : i \in \mathbb{N}\}$ *is bounded for all* $x \in X$ *and any sequence* $\{t_i\} \subset]0, \infty[$, $t_i \searrow 0$.

Then f has a unique fixed point $x^* \in X$ and for each $x_0 \in X$ the sequence $\{f^n(x_0)\}$ converges to x^*.

In [9], Wardowski proposed the question $(Q2)$: Is it possible to omit condition (a) in the last theorem?

Notice that, V. Gregori and J.J. Miñana [3] have shown recently that the class of fuzzy $\mathcal{H}$-contractive mappings is included in the class of fuzzy ψ-contractive mappings.

In this paper we answer in affirmative way the question $(Q1)$ for the (more general) class of fuzzy ψ-contractive mappings when M is strong (Lemma 3.12) or M satisfies $\bigwedge_{t>0} M(x, y, t) > 0$ for each $x, y \in X$ (Corollary 3.8). Then, we state our fuzzy fixed point theorem (Theorem 3.3). As a consequence we answer in affirmative way the question $(Q2)$ and, moreover, we show that the condition (b) in the above theorem can also be omitted (Corollary 3.6). Also, as a consequence of our Lemma 3.12 we deduce a fixed point theorem due to D. Mihet (Theorem 3.13).

The structure of the paper is as follows. After a preliminaries' section, we give our main results in Section 3, which we have mentioned in the last paragraph.

2. Preliminaries

Definition 2.1. (See George and Veeramani [1].) A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a (non-empty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times]0, \infty[$ satisfying the following conditions, for all $x, y, z \in X$, $s, t > 0$:

(GV1) $M(x, y, t) > 0$;
(GV2) $M(x, y, t) = 1$ if and only if $x = y$;
(GV3) $M(x, y, t) = M(y, x, t)$;
(GV4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
(GV5) $M(x, y, \_) :]0, \infty[\to]0, 1]$ is continuous.

If $(X, M, *)$ is a fuzzy metric space, we will say that $(M, *)$ (or simply M) is a *fuzzy metric* on X.

The next definition of *KM*-fuzzy metric space is the reformulation due to Grabiec of the original definition of Kramosil and Michalek [6], which is commonly used by several authors.

Definition 2.2. (See Grabiec [2].) A *KM*-fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a (non-empty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times [0, \infty[$ that satisfies (GV3) and (GV4), and (GV1), (GV2), (GV5) are replaced by (KM1), (KM2), (KM5), respectively, below:

(KM1) $M(x, y, 0) = 0$;
(KM2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$;
(KM5) $M(x, y, \_) : [0, \infty[\to [0, 1]$ is left continuous.

Remark 2.3. $M(x, y, \_)$ is non-decreasing for all $x, y \in X$.

George and Veeramani proved in [1] that every fuzzy metric M on X generates a topology τ_M on X which has as a base the family of open sets of the form $\{B_M(x, \epsilon, t) : x \in X, 0 < \epsilon < 1, t > 0\}$, where $B_M(x, \epsilon, t) = \{y \in X : M(x, y, t) > 1 - \epsilon\}$ for all $x \in X$, $\epsilon \in]0, 1[$ and $t > 0$.