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$\mathcal{Q}$-closure spaces

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Abstract

For a small quantaloid $\mathcal{Q}$, a $\mathcal{Q}$-closure space is a small category enriched in $\mathcal{Q}$ equipped with a closure operator on its presheaf category. We investigate $\mathcal{Q}$-closure spaces systematically with specific attention paid to their morphisms and, as preordered fuzzy sets are a special kind of quantaloid-enriched categories, in particular fuzzy closure spaces on fuzzy sets are introduced as an example. By constructing continuous relations that naturally generalize continuous maps, it is shown (in the generality of the $\mathcal{Q}$-version) that the category of closure spaces and closed continuous relations is equivalent to the category of complete lattices and sup-preserving maps.

Keywords: Quantaloid; $\mathcal{Q}$-closure space; Continuous $\mathcal{Q}$-functor; Continuous $\mathcal{Q}$-distributor; Continuous $\mathcal{Q}$-relation; Complete $\mathcal{Q}$-category; Fuzzy closure space; Fuzzy powerset

1. Introduction

A closure space consists of a (crisp) set X and a closure operator c on the powerset of X; that is, a monotone map $c : \mathbf{2}^X \longrightarrow \mathbf{2}^X$ with respect to the inclusion order of subsets such that $A \subseteq c(A)$, $cc(A) = c(A)$ for all $A \subseteq X$. However, c may not satisfy $c(\varnothing) = \varnothing$, $c(A) \cup c(B) = c(A \cup B)$ for all $A, B \subseteq X$ that are necessary to make itself a *topological* closure operator. The category **Cls** has closure spaces as objects and continuous maps as morphisms, where a map $f : (Y, d) \longrightarrow (X, c)$ between closure spaces is continuous if

$$f^{\rightarrow} d(B) \subseteq c f^{\rightarrow}(B)$$

for all $B \subseteq Y$. It is not difficult to observe that the continuity of the map f is completely determined by its cograph

$$f^{\natural} : X {\multimap\!\!\!\rightarrow} Y, \quad f^{\natural} = \{(x, y) \in X \times Y \mid x = fy\}$$

which must satisfy

$$(f^{\natural})^* d(B) \subseteq c (f^{\natural})^*(B)$$

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for all $B \subseteq Y$; the notion of *continuous relations* then comes out naturally by replacing the cograph $f^\natural$ with a general relation $\zeta : X \looparrowright Y$ (i.e., $\zeta \subseteq X \times Y$) satisfying

$$\zeta^* d(B) \subseteq c\zeta^*(B)$$

for all $B \subseteq Y$, where ζ^* is part of the *Kan adjunction* $\zeta^* \dashv \zeta_*$ induced by ζ [28]:

$$\zeta^*(B) = \{x \in X \mid \exists y \in B : (x, y) \in \zeta\}.$$

The category **ClsRel** of closure spaces and continuous relations admits a natural quotient category $\mathbf{ClsRel}_{\mathsf{cl}}$ of closure spaces and *closed* continuous relations, where a continuous relation $\zeta : (X, c) \looparrowright (Y, d)$ is *closed* if

$$\widetilde{\zeta} y := \{x \in X \mid (x, y) \in \zeta\}$$

is a closed subset of (X, c) for all $y \in Y$. It will be shown that $\mathbf{ClsRel}_{\mathsf{cl}}$ is equivalent to the category **Sup** of complete lattices and sup-preserving maps (Corollary 4.4.3):

$$\mathbf{ClsRel}_{\mathsf{cl}} \simeq \mathbf{Sup}. \tag{1.1}$$

The construction stated above will be explored in a much more general setting in this paper for $\mathcal{Q}$-closure spaces, where $\mathcal{Q}$ is a small *quantaloid*. The theory of quantaloid-enriched categories was initiated by Walters [38], established by Rosenthal [25] and mainly developed in Stubbe's works [32,33]. Based on the fruitful results of quantaloid-enriched categories, recent works of Höhle–Kubiak [10] and Pu–Zhang [22] have established the theory of *preordered fuzzy sets* through categories enriched in a quantaloid $\mathcal{D}Q$ induced by a divisible unital quantale Q. The survey paper [34] is particularly recommended as an overview of this theory for the readership of fuzzy logicians and fuzzy set theorists.

Given a small quantaloid $\mathcal{Q}$, a $\mathcal{Q}$-closure space [28] is a small $\mathcal{Q}$-category (i.e., a small category enriched in $\mathcal{Q}$) $\mathbb{X}$ equipped with a $\mathcal{Q}$-closure operator $c : \mathsf{P}\mathbb{X} \longrightarrow \mathsf{P}\mathbb{X}$ on the presheaf $\mathcal{Q}$-category of $\mathbb{X}$. Before presenting a general form of the categorical equivalence (1.1), we investigate $\mathcal{Q}$-closure spaces systematically with specific attention paid to their morphisms: continuous $\mathcal{Q}$-functors, continuous $\mathcal{Q}$-distributors and finally, closed continuous $\mathcal{Q}$-distributors.

Without assuming a high level of expertise by the readers on quantaloids, we recall in Section 2 the basic notions and techniques of quantaloid-enriched categories that will be employed later. Next, Section 3 is devoted to the study of the category $\mathcal{Q}$-**CatCls** of $\mathcal{Q}$-closure spaces and continuous $\mathcal{Q}$-functors. Explicitly, a $\mathcal{Q}$-functor $f : (\mathbb{X}, c) \longrightarrow (\mathbb{Y}, d)$ between $\mathcal{Q}$-closure spaces is *continuous* if

$$f^{\rightarrow} c \leq d f^{\rightarrow} : \mathsf{P}\mathbb{X} \longrightarrow \mathsf{P}\mathbb{Y}$$

with respect to the pointwise underlying preorder of $\mathcal{Q}$-categories. We also derive a conceptual definition of the specialization (pre)order in a general setting as specialization $\mathcal{Q}$-categories, which has the potential to go far beyond its use in this paper (see Remark 3.3.9).

The main result of this paper is presented in Section 4, where we extend continuous $\mathcal{Q}$-functors to *continuous $\mathcal{Q}$-distributors* as morphisms of $\mathcal{Q}$-closure spaces; that is, $\mathcal{Q}$-distributors $\zeta : (\mathbb{X}, c) \looparrowright (\mathbb{Y}, d)$ between $\mathcal{Q}$-closure spaces with

$$\zeta^* d \leq c\zeta^* : \mathsf{P}\mathbb{Y} \longrightarrow \mathsf{P}\mathbb{X}.$$

The resulting category, $\mathcal{Q}$-**ClsDist**, admits a quotient category $(\mathcal{Q}\text{-}\mathbf{ClsDist})_{\mathsf{cl}}$ of $\mathcal{Q}$-closure spaces and *closed* continuous $\mathcal{Q}$-distributors, where a continuous $\mathcal{Q}$-distributor $\zeta : (\mathbb{X}, c) \looparrowright (\mathbb{Y}, d)$ is *closed* if its transpose

$$\widetilde{\zeta} : \mathbb{Y} \longrightarrow \mathsf{P}\mathbb{X}$$

sends every object y of $\mathbb{Y}$ to a closed presheaf of $(\mathbb{X}, c)$. Although the assignment

$$(\mathbb{X}, c) \mapsto \mathsf{C}(\mathbb{X}, c) \tag{1.2}$$

sending a $\mathcal{Q}$-closure space $(\mathbb{X}, c)$ to the complete $\mathcal{Q}$-category $\mathsf{C}(\mathbb{X}, c)$ of closed presheaves only yields a left adjoint functor from $\mathcal{Q}$-**CatCls** to the category $\mathcal{Q}$-**Sup** of complete $\mathcal{Q}$-categories and sup-preserving $\mathcal{Q}$-functors (Theorem 3.4.3), a little surprisingly, the same assignment (1.2) on objects gives rise to an equivalence of categories (Theorem 4.3.7)

$$(\mathcal{Q}\text{-}\mathbf{ClsDist})_{\mathsf{cl}}^{\mathrm{op}} \simeq \mathcal{Q}\text{-}\mathbf{Sup}, \tag{1.3}$$

which reduces to the equivalence (1.1) when $\mathcal{Q} = \mathbf{2}$.