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Connectedness and local connectedness for lattice-valued convergence spaces

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Abstract

We generalize Preuss' $\mathbb{E}$-connectedness to lattice-valued convergence spaces and prove the basic theory for connected sets, including the product theorem. We further give a suitable definition of local $\mathbb{E}$-connectedness and study its properties.

Keywords: L-topology; L-convergence space; Connectedness; Local connectedness

1. Introduction

Connectedness of topological spaces, resp. of subsets of topological spaces, was defined and studied by F. Hausdorff [8] already in 1914. Much of the theory remained unchanged for about 50 years after that (see e.g. [3]), until in the 70's of the last century G. Preuss recognized that the characterization of a connected space by continuous mappings into a two-point discrete space being constant, lends itself to a useful and broader generalization, which he called $\mathbb{E}$-connectedness [25,26].

In the realm of convergence spaces, connectedness was considered in the dissertation of B.V. Hearsey [9] but also in the textbook of W. Gähler [6] we find some results. Hearsey left open the question if connectedness was preserved by taking products also for convergence spaces. This was resolved in the affirmative by R. Vainio [29], who, in the sequel, applied Preuss' concept of $\mathbb{E}$-connectedness to various types of convergence spaces [30–32].

In lattice-valued topology, right from the beginning, connectedness was studied. There are several ways of generalizing the classical definition and different concepts resulted, see e.g. [12,23,27,28]. Lowen and Srivastava applied Preuss' connectedness concept to stratified [0, 1]-topological spaces [24]. Also local connectedness was studied in lattice-valued topological spaces, see e.g. [20].

In this paper, we will work in the realm of lattice-valued convergence spaces, where the lattice L is given by a complete Heyting algebra [14,15]. We apply Preuss' connectedness concept here and prove all basic results, including

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the product theorem. We further give a suitable definition of local $\mathbb{E}$-connectedness, generalizing a corresponding definition from convergence spaces (see e.g. [6]) and show that the important properties, like preservation under final constructions, under open subspaces or the productivity, are valid with our definition.

The paper is organised as follows. We first collect, in Section 2, the necessary theory on lattices, lattice-valued filters and lattice-valued convergence spaces. Section 3 then treats two connectedness concepts for lattice-valued convergence spaces, while the next section generalizes these to Preuss' concept of $\mathbb{E}$-connectedness. Section 5 then studies the properties of $\mathbb{E}$-connected subsets. Local $\mathbb{E}$-connectedness is defined and studied in Section 6. Finally we draw some conclusions.

2. Preliminaries

We consider in this paper *frames*, i.e. complete lattices L (with bottom element $\bot$ and top element $\top$) for which the infinite distributive law $\bigvee_{j\in J}(\alpha\wedge\beta_j)=\alpha\wedge\bigvee_{j\in J}\beta_j$ holds for all $\alpha,\beta_j\in L$ $(j\in J)$. In a frame L, we can define an implication operator by $\alpha\to\beta=\bigvee\{\gamma\in L\,:\,\alpha\wedge\gamma\le\beta\}$. This implication is then right-adjoint to the meet operation, i.e. we have $\delta\le\alpha\to\beta$ iff $\alpha\wedge\delta\le\beta$. A complete lattice L is *completely distributive* if the following distributive laws are true.

$$\text{(CD1)}\quad \bigvee_{j\in J}\left(\bigwedge_{i\in I_j}\alpha_{ji}\right)=\bigwedge_{f\in\prod_{j\in J}I_j}\left(\bigvee_{j\in J}\alpha_{jf(i)}\right)$$

$$\text{(CD2)}\quad \bigwedge_{j\in J}\left(\bigvee_{i\in I_j}\alpha_{ji}\right)=\bigvee_{f\in\prod_{j\in J}I_j}\left(\bigwedge_{j\in J}\alpha_{jf(i)}\right)$$

It is well known that, in a complete lattice, (CD1) if and only if (CD2). In any complete lattice we can define the *wedge-below relation* $\alpha\lhd\beta$ if for all subsets $D\subseteq L$ such that $\beta\le\bigvee D$ there is $\delta\in D$ such that $\alpha\le\delta$. Then $\alpha\le\beta$ whenever $\alpha\lhd\beta$ and $\alpha\lhd\bigvee_{j\in J}\beta_j$ iff $\alpha\lhd\beta_i$ for some $i\in J$. A complete lattice is completely distributive if and only if we have $\alpha=\bigvee\{\beta\,:\,\beta\lhd\alpha\}$ for any $\alpha\in L$, see e.g. Theorem 7.2.3 in [1]. An element $\alpha\in L$ in a lattice is called *prime* if $\beta\wedge\gamma\le\alpha$ implies $\beta\le\alpha$ or $\gamma\le\alpha$. For more results on lattices we refer to [7].

For notions from category theory, we refer to the textbook [2].

For a frame L and a set X, we denote the set of all L-sets $a,b,c,\ldots:X\longrightarrow L$ by L^X. We define, for $\alpha\in L$ and $A\subseteq X$, the L-set α_A by $\alpha_A(x)=\alpha$ if $x\in A$ and $=\bot$ else. In particular, we denote the constant L-set with value $\alpha\in L$ by α_X and $\top_A$ is the characteristic function of $A\subseteq X$. The operations and the order are extended pointwisely from L to L^X.

A mapping $\mathcal{F}:L^X\longrightarrow L$ is called a *stratified L-filter on X* [11] if (F1) $\mathcal{F}(\top_X)=\top$ and $\mathcal{F}(\bot_X)=\bot$, (F2) $\mathcal{F}(a)\le\mathcal{F}(b)$ whenever $a\le b$, (F3) $\mathcal{F}(a)\wedge\mathcal{F}(b)\le\mathcal{F}(a\wedge b)$ and (Fs) $\mathcal{F}(\alpha_X)\ge\alpha$ for all $a,b\in L^X$, $\alpha\in L$. A typical example is, for $x\in X$, the *point L-filter* $[x]$ defined by $[x](a)=a(x)$ for all $a\in L^X$. We denote the set of all stratified L-filters on X by $\mathcal{F}_L^s(X)$ and order it by $\mathcal{F}\le\mathcal{G}$ if for all $a\in L^X$ we have $\mathcal{F}(a)\le\mathcal{G}(a)$. If $\mathcal{F}\le\mathcal{G}$, we call $\mathcal{G}$ *finer than* $\mathcal{F}$. Maximal stratified L-filters in this order exist and are called *stratified L-ultrafilters*, and for every stratified L-filter there is a finer stratified L-ultrafilter, cf. [11]. For a family of filters $\mathcal{F}_i$ $(i\in J)$, the infimum in the order is given by $(\bigwedge_{i\in J}\mathcal{F}_i)(a)=\bigwedge_{i\in J}\mathcal{F}_i(a)$ for all $a\in L^X$. The supremum, however, only exists if $\mathcal{F}_{i_1}(a_1)\wedge\mathcal{F}_{i_2}(a_2)\wedge\ldots\wedge\mathcal{F}_{i_n}(a_n)=\bot$ whenever $a_1\wedge a_2\wedge\ldots\wedge a_n=\bot_X$. In this case it is given by $(\bigvee_{i\in J}\mathcal{F}_i)(a)=\bigvee\{\mathcal{F}_{i_1}(a_1)\wedge\mathcal{F}_{i_2}(a_2)\wedge\ldots\wedge\mathcal{F}_{i_n}(a_n)\,:\,a_1\wedge a_2\wedge\ldots\wedge a_n\le a\}$, see [11]. Consider now a mapping $f:X\longrightarrow Y$. For $\mathcal{F}\in\mathcal{F}_L^s(X)$ then $f(\mathcal{F})\in\mathcal{F}_L^s(Y)$ is defined by $f(\mathcal{F})(b)=\mathcal{F}(f^{\leftarrow}(b))$ with $f^{\leftarrow}(b)=b\circ f$ for $b\in L^X$, [11]. For $\mathcal{G}\in\mathcal{F}_L^s(Y)$ we define $f^{\leftarrow}(\mathcal{G})(a)=\bigvee\{\mathcal{G}(b)\,:\,f^{\leftarrow}(b)\le a\}$. If $\mathcal{G}(b)=\bot$ whenever $f^{\leftarrow}(b)=\bot_X$, then $f^{\leftarrow}(\mathcal{G})\in\mathcal{F}_L^s(X)$, see [14]. We will need the following two examples later. Firstly, if $M\subseteq X$ we define $i_M: M\longrightarrow X$, $i_M(x)=x$. In case of existence, we denote, for $\mathcal{F}\in\mathcal{F}_L^s(X)$, $\mathcal{F}_M=i_M^{\leftarrow}(\mathcal{F})$. Secondly, for sets X_i $(i\in J)$, we denote the projections $p_j:\prod_{i\in J}X_i\longrightarrow X_j$ and define the *stratified L-product filter* $\prod_{i\in J}\mathcal{F}_i=\bigvee_{i\in J}p_i^{\leftarrow}(\mathcal{F}_i)$, see [14]. The following result follows directly from the definition.

Lemma 2.1. *Let $\mathcal{F}_i\in\mathcal{F}_L^s(X_i)$ for $i\in J$. Then, for $a\in L^{\prod_{i\in J}X_i}$,*

$$\prod_{i\in J}\mathcal{F}_i(a)=\bigvee\left\{\bigwedge_{i\in J}\mathcal{F}_i(b_i)\,:\,\prod_{i\in J}b_i\le a \text{ and only finitely many } b_i\ne\top_{X_i}\right\}.$$