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Weak monotonicity of Lehmer and Gini means

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Abstract

We analyze directional monotonicity of several mixture functions in the direction $(1, 1 \ldots, 1)$, called weak monotonicity. Our particular focus is on power weighting functions and the special cases of Lehmer and Gini means. We establish limits on the number of arguments of these means for which they are weakly monotone. These bounds significantly improve the earlier results and hence increase the range of applicability of Gini and Lehmer means. We also discuss the case of affine weighting functions and find the smallest constant which ensures weak monotonicity of such mixture functions.

Keywords: Mixture function; Weak monotonicity; Lehmer mean; Gini mean

1. Introduction

Aggregation functions play an important role in many applications including decision making, fuzzy systems and image processing [8,19,27]. In particular, aggregation functions are suitable models for fuzzy connectives [16]. Averaging functions, whose prototypical examples are the arithmetic mean and the median, allow compensation between low values of some inputs and high values of the others. These functions are often used in some fuzzy logic systems [35], for example in the weighted compensative logic [17]. More sophisticated averaging functions in which the weightings of the inputs depend on their magnitude include mixture functions [21,25], density based means [1], statistically grounded aggregation operators [23] and Bajraktarević means, which generalize quasi-arithmetic means [2,3].

Such functions with variable weights are not necessarily monotone in all arguments, and technically do not fit the definition of the aggregation functions. Some sufficient conditions for monotonicity of mixture functions were established in [22]. A recently proposed concept of weak monotonicity [9,13,32–34] presumes that the value of the aggregate does not decrease when all the inputs are increased by the same value, but may actually decrease if a sub-

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set of the inputs increases. For example, when combining the consequents of the If–Then rules in fuzzy rule-based systems, we may want to discard some of the activated rules if their output is very inconsistent with the rest. Weak monotonicity is very useful when calculating representative values of clusters of data in the presence of outliers. Indeed, cluster structure may change when only some inputs are increased (or decreased), but it does not change when all inputs are changed by the same value. It was shown that robust location estimators [26] and some useful classes of means, like Lehmer means, certain mixture functions, density-based means and the mode, are all weakly monotone [9,13,32–34]. Recently the notion of pre-aggregation functions was proposed as a generalization of aggregation functions [20], where the functions are not necessarily monotone but directionally monotone. The weakly monotone functions are hence pre-aggregation functions.

It becomes important to establish conditions under which various averaging functions are weakly monotone. In this paper, we will deal with some mixture and quasi-mixture functions, and in particular with Lehmer and Gini means. In mixture (and quasi-mixture) functions the inputs are averaged as in a weighted mean, but the weights depend on the inputs. Weights can thus be chosen so as to alternatively emphasize or de-emphasize the small or large inputs. In Lehmer and Gini means the weighting function is the power function. We improve the results of [9,33] which provide some sufficient conditions of monotonicity of Lehmer means. We also improve the results related to weak monotonicity of Gini means [5,34].

The remainder of this article is structured as follows. In Section 2, we provide the necessary mathematical foundations that underpin aggregation functions and means, which we rely on in subsequent sections. Our main results are concentrated in Sections 3, 4 and 5. In Section 6, we recall duality of mixture functions. Our conclusions are presented in Section 7. The proofs of Theorem 3, Theorem 5 and Theorem 6 are presented in Appendix A.

2. Preliminaries

2.1. Aggregation functions

In this article, we make use of the following notations and assumptions. Without loss of generality, we assume that the domain of interest is any closed, non-empty interval $\mathbb{I} = [a, b] \subseteq \bar{\mathbb{R}} = [-\infty, \infty]$ and that tuples in $\mathbb{I}^n$ are defined as $\mathbf{x} = (x_{i,n} | n \in \mathbb{N}, i \in \{1, \dots, n\})$. We write x_i as the shorthand for $x_{i,n}$ such that it is implicit that $i \in \{1, \dots, n\}$. Furthermore, $\mathbb{I}^n$ is ordered such that for $\mathbf{x}, \mathbf{y} \in \mathbb{I}^n$, $\mathbf{x} \le \mathbf{y}$ implies that each component of $\mathbf{x}$ is no greater than the corresponding component of $\mathbf{y}$. Unless otherwise stated, a constant vector given as $\mathbf{c}$ is taken to mean $\mathbf{c} = c(\underbrace{1, 1, \dots, 1}_{n \text{ times}}) = c\mathbf{1}$, $c > 0$, where n is implicit within the context of use.

Consider now the following definitions, adopted from [8,14,19,27].

Definition 1. A function $F : \mathbb{I}^n \to \bar{\mathbb{R}}$ is **monotone** (increasing) if $\forall\, \mathbf{x}, \mathbf{y} \in \mathbb{I}^n$, $\mathbf{x} \le \mathbf{y}$ then $F(\mathbf{x}) \le F(\mathbf{y})$.

Note that we use the terms *increasing/decreasing* not in the strict sense. When the inequality $F(\mathbf{x}) < F(\mathbf{y})$ is strict and $\mathbf{x}$ is different from $\mathbf{y}$, we will use the terms *strictly* increasing.

For differentiable functions, monotonicity is equivalent to the condition that the directional derivative $D_{\mathbf{e}_i}(F)(\mathbf{x}) = \nabla F(\mathbf{x}) \cdot \mathbf{e}_i \ge 0$ at each point $\mathbf{x} \in \mathbb{I}^n$, for all $i \in \{1, 2, .., n\}$, where vectors $\mathbf{e}_i$ come from the canonical Euclidean basis.

Definition 2. A function $F : \mathbb{I}^n \to \mathbb{I}$ is an **aggregation function** in $\mathbb{I}^n$ if F is monotone increasing in $\mathbb{I}$ and $F(\mathbf{a}) = a$, $F(\mathbf{b}) = b$.

Definition 3. A function $F : \mathbb{I}^n \to \mathbb{I}$ is **idempotent** if for every input $\mathbf{x} = (t, t, \dots, t)$, $t \in \mathbb{I}$ the output is $F(\mathbf{x}) = t$.

Definition 4. A function $F : \mathbb{I}^n \to \mathbb{I}$ has **averaging behavior** (or is averaging) if for every $\mathbf{x}$ it is bounded by $\min(\mathbf{x}) \le F(\mathbf{x}) \le \max(\mathbf{x})$.

Functions that have averaging behavior are idempotent and monotone increasing idempotent functions are averaging. Another term often used synonymously with the average is the *mean*. We will use the term averaging throughout this paper.