Available online at www.sciencedirect.com
ScienceDirect
Fuzzy Sets and Systems 299 (2016) 41–65
FUZZY sets and systems
www.elsevier.com/locate/fss

On distributivity equations for uninorms over semi-t-operators ☆

Yong Su, Wenwen Zong, Hua-Wen Liu *

School of Mathematics, Shandong University, Jinan, Shandong 250100, China

Received 14 March 2015; received in revised form 21 July 2015; accepted 3 August 2015
Available online 10 August 2015

Abstract

Recently, Drygaś generalized nullnorms and t-operators and introduced semi-t-operators by eliminating commutativity from the axioms of t-operators. Distributivity equations were investigated in families of certain operations (e.g. triangular norms, conorms, uninorms and nullnorms (or called t-operators)). In this paper, we give out the solutions of distributivity equations for uninorms over semi-t-operators. Previous results about distributivity equations for uninorms over nullnorms can be obtained as corollaries.

Keywords: Fuzzy connectives and aggregation operators; Functional equations; Distributivity condition; Uninorm; Semi-t-operator

1. Introduction

Distributivity between two operations is a property that was already posed many years ago [1]. This problem is related to the so-called pseudo-analysis, where the structure of $\mathbb{R}$ as a vector space is replaced by the structure of semi-ring on any interval $[a, b] \subseteq [-\infty, +\infty]$, denoting the corresponding operations as pseudo-addition and pseudo-multiplication. In the semi-ring structure and some generalizations, the distributivity plays a fundamental and essential role. In this context, t-norms and t-conorms have been used to model the mentioned pseudo-operations and uninorms have been used as well (see [16,17,25,32]). Hence, there appears a new approach direction, that is, distributivity of t-norms and t-conorms ([11], p. 17), aggregation operators, quasi-arithmetic means [3], Mayor's aggregation operators and semi-uninorms [15], fuzzy implications [2], uninorms and nullnorms (also called t-operators) [9,22,24,34], semi-uninorms and t-conorms (or t-norms) [19], semi-t-operators and semi-nullnorms [7], semi-t-operators [8], semi-t-operators over uninorms [33].

This paper is a continuation of [33] and will explore the other direction, namely, distributivity of uninorms over semi-t-operators. On the other hand, we will extend research from [9,22,24,34] towards the distributivity of uninorms over the semi-t-operators (namely, non-commutative t-operators). Previous results about distributivity equations for

☆ Supported by the National Natural Science Foundation of China (No. 61174099) and the Research Found for the Doctoral Program of Higher Education of China (No. 20120131110001).
* Corresponding author.
E-mail addresses: yongsu88@163.com (Y. Su), zongwen198811@163.com (W. Zong), hw.liu@sdu.edu.cn (H.-W. Liu).

http://dx.doi.org/10.1016/j.fss.2015.08.001
0165-0114/

uninorms over nullnorms are one special case, i.e. when semi-t-operators are commutative. We follow in this way and in particular we want to investigate the distributivity equations

$$G_1(x, G_2(y, z)) = G_2(G_1(x, y), G_1(x, z)) \tag{1}$$

$$G_1(G_2(y, z), x) = G_2(G_1(y, x), G_1(z, x)) \tag{2}$$

where the unknown functions G_1 and G_2 are, respectively, an uninorm and a semi-t-operator and hence differ from the ones in [9,22,24,33,34].

Thus, the goal of this paper is to show new results and to generalize those already known about the distributivity of uninorms over nullnorms. Moreover, this paper only deals with the distributivity condition among different classes of operators, and does not address uninorms only. In this sense, it differs from many other subsequent papers dealing with distributivity between uninorms (see [27,29,30]).

The paper is organized as follows. First, we review some concepts and results about uninorms, t-operators and semi-t-operators (Section 2). Then, solutions of distributivity equations from described families are characterized (Section 3).

2. Preliminaries

We assume that the reader is familiar with the results concerning t-norms, t-conorms, nullnorms and t-operators (see [4,16,21,22]). To make this work self-contained, we recall some concepts and results used in the rest of the paper.

Definition 1. (See [7].) (i) A semi-t-norm T (semi-t-conorm S) on $[0, 1]$ is an increasing, associative operation with neutral element 1 (neutral element 0).

(ii) A t-norm T (t-conorm S) is a commutative semi-t-norm (semi-t-conorm).

Remark 1. In [10,18], a **semi**-t-norm (**semi**-t-conorm) in Definition 1 is called a pseudo-t-norm (pseudo-t-conorm) and a t-**semi**norm (t-**semi**conorm) is a non-commutative and non-associative t-norm (t-conorm). In this paper, we uniformly use the terminology from [7] for the sake of consistency.

Definition 2. (See [7].) An operation $F : [0, 1]^2 \to [0, 1]$ is called semi-t-operator if it is associative, increasing, fulfill $F(0, 0) = 0$, $F(1, 1) = 1$ and such that the functions F_0, F_1, F^0, F^1 are continuous, where $F_0(x) = F(0, x)$, $F_1(x) = F(1, x)$, $F^0(x) = F(x, 0)$ and $F^1(x) = F(x, 1)$.

Denote $\mathcal{F}_{a,b}$ the family of all semi-t-operators, such that $F(0, 1) = a$ and $F(1, 0) = b$.

Theorem 1. *(See [7].) $F \in \mathcal{F}_{a,b}$ iff there exist semi-t-norm T and semi-t-conorm S such that*

$$F(x, y) = \begin{cases} aS(\frac{x}{a}, \frac{y}{a}) & \text{if } x, y \in [0, a], \\ b + (1-b)T(\frac{x-b}{1-b}, \frac{y-b}{1-b}) & \text{if } x, y \in [b, 1], \\ a & \text{if } x \le a \le y, \\ b & \text{if } y \le b \le x, \\ x & \text{if } a \le x \le b, \end{cases} \tag{3}$$

when $a \le b$ and

$$F(x, y) = \begin{cases} bS(\frac{x}{b}, \frac{y}{b}) & \text{if } x, y \in [0, b], \\ a + (1-a)T(\frac{x-a}{1-a}, \frac{y-a}{1-a}) & \text{if } x, y \in [a, 1], \\ a & \text{if } x \le a \le y, \\ b & \text{if } y \le b \le x, \\ y & \text{if } b \le y \le a, \end{cases} \tag{4}$$

when $b \le a$.

Drygaś [7] pointed out that the class of nullnorms (also called t-operators) is a subclass of semi-t-operators.