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A note on the superadditive and the subadditive transformations of aggregation functions

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Abstract

We expand the theoretical background of the recently introduced superadditive and subadditive transformations of aggregation functions. Necessary and sufficient conditions ensuring that a transformation of a proper aggregation function is again proper are deeply studied and exemplified. Relationships between these transformations are also studied.

Keywords: Aggregation function; Subadditive transformation; Superadditive transformation

1. Introduction

Motivated by applications in economics, subadditive and superadditive transformations of aggregation functions on $R^+ = [0, \infty[$ have been recently introduced in [4]. Formally, both these transformations can be introduced on the improper real interval $[0, \infty]$.

Definition 1. A mapping $A : [0, \infty]^n \to [0, \infty]$ is called an (n-ary) aggregation function if $A(0, \dots, 0) = 0$ and A is increasing in each coordinate. Further, A is called a proper (n-ary) aggregation function if it satisfies the following two additional constraints:

(i) $A(\mathbf{x}) \in]0, \infty[$ for some $\mathbf{x} \in]0, \infty[^n$,
(ii) $A(\mathbf{x}) < \infty$ for all $\mathbf{x} \in [0, \infty[^n$.

Though for real applications we only need proper aggregation functions (in fact, their restriction to the domain $[0, \infty[^n$), a broader framework of all (n-ary) aggregation functions is of advantage in a formal description of our results, making formulations and expressions more transparent. Observe that our framework is broader than the concept of aggregation functions on $[0, \infty]$ as introduced in [1,3], which does not cover Sugeno integral based aggregation

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functions, for example. We denote the class of all n-ary aggregation functions by $\mathcal{A}_n$, and the class of all n-ary proper aggregation functions by $\mathcal{P}_n$.

The next definition was motivated by optimization tasks treated in linear programming area and related areas [2], as well as by recently introduced concepts of concave [5] and convex [6] integrals.

Definition 2. For every $A \in \mathcal{A}_n$ the subadditive transformation $A_* : [0, \infty]^n \to [0, \infty]$ of A is given by

$$A_*(\mathbf{x}) = \inf \{\sum_{i=1}^{k} A(\mathbf{y}^{(i)}) \mid \sum_{i=1}^{k} \mathbf{y}^{(i)} \geq \mathbf{x}\} \tag{1}$$

Similarly, for every $A \in \mathcal{A}_n$ the superadditive transformation $A^* : [0, \infty]^n \to [0, \infty]$ of A is defined by

$$A^*(\mathbf{x}) = \sup \{\sum_{j=1}^{\ell} A(\mathbf{y}^{(j)}) \mid \sum_{j=1}^{\ell} \mathbf{y}^{(j)} \leq \mathbf{x}\} . \tag{2}$$

Observe that the transformation (1) was originally introduced in [4] for $A \in \mathcal{K}_*^n$, where $\mathcal{K}_*^n$ is the class of all n-ary proper aggregation functions (restricted to $[0, \infty[^n$) such that also A_* is proper, that is, $A_* \in \mathcal{P}_n$. Similarly, A^* given by (2) was originally introduced in [4] only for $A \in \mathcal{K}_n^*$, where $\mathcal{K}_n^*$ is the class of all $A \in \mathcal{P}_n$ (restricted to $[0, \infty[^n$), so that $A^* \in \mathcal{P}_n$ as well.

Theorem 2 in [4] gives a necessary and sufficient condition ensuring that a function $A \in \mathcal{P}_n$ has also the property that $A \in \mathcal{K}_n^*$. We develop this result, giving an equivalent condition. Moreover, we also characterize all the functions $A \in \mathcal{P}_n$ such that $A \in \mathcal{K}_*^n$. Our approach is based on a deep study of transformations (1) and (2) on unary aggregation functions that belong to $\mathcal{P}_1$. Our approach allows to show that for any $A \in \mathcal{P}_n$ we have the inequality $(A_*)^* \leq (A^*)_*$.

The paper is organized as follows. In the next section, the classes $\mathcal{K}_1^*$ and $\mathcal{K}_*^1$ are completely described, showing that the properties in a neighborhood of 0 are important for characterization of elements of these classes. In Section 3, necessary and sufficient conditions for a function $A \in \mathcal{P}_n$ to belong to $\mathcal{K}_n^*$, or to $\mathcal{K}_*^n$, are given. Section 4 is devoted to the study of relationships of transformations $(A_*)^*$ and $(A^*)_*$. Finally, some concluding remarks are added.

2. The one-dimensional case

We begin with basic results which show how the values of the subadditive and superadditive transformations of one-dimensional aggregation functions depend on the behavior of the functions near zero.

Theorem 1. *Let h be an unary aggregation function on $[0, \infty]$ with $\liminf_{t\to 0^+} h(t)/t = a$ and $\limsup_{t\to 0^+} h(t)/t = b$, where $0 \leq a \leq b \leq \infty$. Then, for every $x \in]0, \infty[$ we have $h_*(x) \leq ax$ and $h^*(x) \geq bx$.*

Proof. Let $x > 0$. By definitions of h_* and h^*, for every positive integer n we have $h_*(x) \leq nh(x/n) \leq h^*(x)$, that is,

$$h_*(x) \leq x \cdot \frac{h(\frac{x}{n})}{\frac{x}{n}} \leq h^*(x) . \tag{3}$$

Since h is increasing, for every t such that $\frac{x}{n+1} \leq t \leq \frac{x}{n}$ we have

$$\frac{h(\frac{x}{n+1})}{\frac{x}{n}} \leq \frac{h(t)}{t} \leq \frac{h(\frac{x}{n})}{\frac{x}{n+1}} .$$

Applying the limits inferior and superior to these inequalities as $t \to 0^+$ and $n \to \infty$ (with $\frac{n+1}{n} \to 1$) shows that

$$\liminf_{n\to\infty} \frac{h(\frac{x}{n})}{\frac{x}{n}} \leq \liminf_{t\to 0^+} \frac{h(t)}{t} \quad \text{and} \quad \limsup_{n\to\infty} \frac{h(\frac{x}{n})}{\frac{x}{n}} \geq \limsup_{t\to 0^+} \frac{h(t)}{t} . \tag{4}$$

Combining (3) with (4) now gives

$$h_*(x) \leq x \cdot \liminf_{t\to 0^+} \frac{h(t)}{t} = ax \quad \text{and} \quad h^*(x) \geq x \cdot \limsup_{t\to 0^+} \frac{h(t)}{t} = bx$$

for every $x > 0$, which completes the proof. □