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T-norms and t-conorms continuous around diagonals

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Abstract

Triangular norms and conorms with continuous diagonals are discussed. In literature we can find examples of non-continuous t-norms with continuous diagonals, however, their deeper study is still missing. In this paper we introduce a sufficient condition under which a t-norm with continuous diagonal is continuous. Moreover, we show that an Archimedean t-norm continuous on the boundary is continuous. Several illustrative examples are also included.

Keywords: t-Norm; t-Conorm; Ordinal sum

1. Introduction and basic notions

Triangular norms and conorms [1,5] are applied in many domains and therefore knowledge of the structure of the class of t-norms (t-conorms) is very important. An open question whether a t-norm with continuous diagonal is continuous was answered negatively in [7,9,11]. In this paper we introduce a sufficient condition under which a t-norm with continuous diagonal is necessarily continuous. First, we introduce some basic notions.

A triangular norm (see [1,5,8,10]) is a function $T\colon [0,1]^2 \longrightarrow [0,1]$ which is commutative, associative, non-decreasing in both variables and 1 is its neutral element. Due to the associativity, n-ary form of any t-norm is uniquely given and thus it can be extended to an aggregation function (see [2]) acting on $\bigcup_{n\in\mathbb{N}}[0,1]^n$. In some proofs we will use a ternary form of a t-norm instead of binary, if it is suitable. Dual functions to t-norms are t-conorms. A triangular conorm is a function $S\colon [0,1]^2 \longrightarrow [0,1]$ which is commutative, associative, non-decreasing in both variables and 0 is its neutral element. The duality between t-norms and t-conorms is expressed by the fact that from any t-norm T we can obtain its dual t-conorm S by the equation

$$S(x,y) = 1 - T(1-x, 1-y)$$

and vice versa. Thus all results that we show for t-norms can be immediately obtained also for t-conorms. This is the reason why we will show all results only for t-norms.

Let us recall the ordinal sum construction for t-norms.

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Proposition 1. *(See [5].) Let K be a finite or countably infinite index set and let $(]a_k, b_k[)_{k\in K}$ be a disjoint system of open subintervals of* $[0, 1]$. *Let $(T_k)_{k\in K}$ be a system of t-norms. Then the ordinal sum $T = (\langle a_k, b_k, T_k\rangle \mid k \in K)$ given by*

$$T(x,y) = \begin{cases} a_k + (b_k - a_k)T_k(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}) & \text{if } (x,y) \in [a_k, b_k[^2, \\ \min(x,y) & \text{else} \end{cases}$$

is a t-norm. The t-norm T is continuous if and only if all summands T_k for $k \in K$ are continuous.

Definition 1.

(i) A diagonal of a t-norm T is a function $d_T : [0,1] \longrightarrow [0,1]$ given by $d_T(x) = T(x,x)$ for all $x \in [0,1]$.
(ii) A t-norm $T : [0,1]^2 \longrightarrow [0,1]$ is called Archimedean if for any fixed couple $(x,y) \in]0,1[^2$ there exists an $n \in \mathbb{N}$ such that $y_T^{(n)} < x$, where $y_T^{(n)} = T(\underbrace{y, \dots, y}_{n\text{-times}})$. Equivalently, a t-norm T is Archimedean if for all $z \in [0,1[$ we have $\lim\limits_{n\longrightarrow\infty} z_T^{(n)} = 0$.

Definition 2. (See [12].) A binary function $T : [0,1]^2 \longrightarrow [0,1]$ which is commutative, associative, non-decreasing in both variables, $T(1,1) = 1$ and $T(x,y) \le \min(x,y)$ for all $(x,y) \in [0,1]^2$ is called a boundary weak t-norm (bwt-norm for short).

Similarly as in the case of t-norms, a bwt-norm T is Archimedean if for all $z \in [0,1[$ we have $\lim\limits_{n\longrightarrow\infty} z_T^{(n)} = 0$.

Definition 3. For a given binary operation $O : [0,1]^2 \longrightarrow [0,1]$ and each $x \in [0,1]$ we define a function $o_x : [0,1] \longrightarrow [0,1]$ by $o_x(z) = O(x,z)$ for $z \in [0,1]$.

2. Triangular norms with continuous diagonals

Several examples of non-continuous t-norms with continuous diagonals can be found in [7,9,11].

Example 1. For every $x \in [0,1]$ assume its triadic expansion $(x_n)_{n\in\mathbb{N}}$, i.e., $x = \sum\limits_{n\in\mathbb{N}} \frac{x_n}{3^n}$, where $x_n \in \{0,1,2\}$ for all $n \in \mathbb{N}$. Then 0 corresponds to the expansion where $x_n = 0$ for all $n \in \mathbb{N}$, and 1 corresponds to the expansion where $x_n = 2$ for all $n \in \mathbb{N}$. The set of pure Cantor points is the set C such that each $x \in C$ has a triadic expansion containing only 0 and 2. Let S be the set of points $x \in [0,1]$ such that for the triadic expansion $(x_n)_{n\in\mathbb{N}}$ of every point $x \in S$ there exists an $N \in \mathbb{N}$ such that $x_p \in \{0,2\}$ for $p < N$, $x_N = 1$ and $x_p = 0$ for $p > N$. Note that the set S is the set of left Cantor points and if $x \in S$ then $x = \frac{i}{3^j}$ for some $i, j \in \mathbb{N}$ (see [9]). Further, each point $x \in S$ will be represented, for the corresponding $N \in \mathbb{N}$, by the sequence $x_1, \dots, x_{N-1}$, i.e., $x \approx (x_1, \dots, x_{N-1})$. Recall that such a representation contains only 0 and 2. Let $f : S \longrightarrow [0,\infty]$ be a function for which there is $f(\emptyset) = 1$, $f((0, x_1, \dots, x_p)) = 1 + f((x_1, \dots, x_p))$, and $f((x_1, \dots, x_p)) = \frac{1}{f((2-x_1, \dots, 2-x_p))}$. Then f is uniquely given by these properties (see [9]). For each $p \in S$, where $p = \frac{i}{3^j}$, we further define an increasing linear function $g_p : \left[\frac{1}{3}, \frac{2}{3}\right] \longrightarrow \left[\frac{i}{3^j}, \frac{i+1}{3^j}\right]$. We define a binary operation $T_K : [0,1]^2 \longrightarrow [0,1]$ by

$$T_K(x,y) = \begin{cases} 0 & \text{if } \min(x,y) = 0, \\ \min(x,y) & \text{if } \max(x,y) = 1, \\ g_{p+q}(\min(g_p^{-1}(x), g_q^{-1}(y))) & \text{if } x \in \left[\frac{i}{3^j}, \frac{i+1}{3^j}\right], y \in \left[\frac{m}{3^n}, \frac{m+1}{3^n}\right], \\ & p = f(\frac{i}{3^j}), q = f(\frac{m}{3^n}), \\ \inf\{T(c,d) \mid c \in S, d \in S, x \le c, y \le d\} & \text{otherwise.} \end{cases}$$