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Equivalence operators in nilpotent systems

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Abstract

A consistent connective system generated by nilpotent operators is not necessarily isomorphic to the Łukasiewicz system. Using more than one generator function, consistent nilpotent connective systems (so-called bounded systems) can be obtained with the advantage of three naturally derived negation operators and thresholds. In this paper, equivalences in bounded systems are examined. Here, three different types of operators are studied, and a paradox of the equivalence (i.e. there is no equivalence relation in a non-Boolean setting which fulfils $\forall x\ e(x, x) = 1$ and $e(x, n(x)) = 0$) is resolved by aggregating the implication-based equivalence and its dual operator. We will also show that the aggregated equivalence has nice properties like associativity, threshold transitivity and T-transitiviy.

Keywords: Equivalence; Nilpotent operator; Łukasiewicz operator system; Bounded system; T-transitivity; Associativity

1. Introduction

The theory of fuzzy relations is a generalization of that of crisp relations of a set. Zadeh introduced the concept of fuzzy relations in [28] and the concept of fuzzy similarity relations in [29]. Since then, many authors studied fuzzy equivalence relations [6,7,22,23] and it has proven to be useful in different contexts such as fuzzy control, approximate reasoning and fuzzy cluster analysis.

As shown by Gupta and Gupta [18], the condition $\mu(x, x) = 1$ for $\forall x \in X$ is too strong for defining a fuzzy reflexive relation μ on a set X (see also [27] and [8]). Therefore, new types of fuzzy reflexive relations were needed to be introduced. In [27], the concepts of ϵ-reflexive fuzzy relations and weakly reflexive fuzzy relations were defined by weakening the standard reflexive fuzzy relation to $\mu(x, x) \geq \epsilon > 0$. Gupta and Gupta [18] introduced G-reflexive fuzzy relations as a generalization of reflexive fuzzy relations.

While discussing fuzzy transitive relations, different approaches have been adopted. The first type of transitivity is that introduced by Zadeh [29], and the second type of transitivity is the so-called T-transitivity of fuzzy relations,

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defined with the help of the t-norm. In [4,5,10,11], fuzzy T-transitivity has been deeply studied. Recently, Mesiar et al. [21] have noticed that the associativity of a t-norm is superfluous in the above context, especially since we never have to aggregate more than two arguments. Thus, they have substituted a conjunctor instead of a t-norm. An alternative approach based on implications has been considered in [25,26]. In [19], I-transitivity, where the implicator I is nothing more than a binary operator satisfying the boundary conditions of an implication, was studied. Another type of transitivity, the so-called ϵ-fuzzy transitivity, has been introduced in [3]. In [1], the authors introduced the concept of (α, β)-fuzzy reflexive relations, as a generalization of fuzzy reflexive relation as well as of fuzzy G-reflexive relations. More general types of fuzzy symmetric relation, a (α, β)-fuzzy symmetric relation and (α, β)-fuzzy transitive relations, were also studied. The concepts of (α, β)-fuzzy reflexive, symmetric and transitive relations naturally lead to the concept of (α, β)-fuzzy equivalence relations on a set. In [9], the concept of a T-partition was introduced as a generalization of that of a classical partition.

Although the mentioned list of authors is by no means complete, it gives us a slight idea about the importance of the concept of fuzzy equivalence relations in different contexts. In our work we resolve a paradox of the equivalence (i.e. there is no equivalence relation in a non-Boolean setting which fulfils $\forall x\ e(x, x) = 1$ and $e(x, n(x)) = 0$) by aggregating the implication-based equivalence and its dual operator.

In our previous article [14], we showed that a consistent connective system generated by nilpotent operators is not necessarily isomorphic to the Łukasiewicz system. Using more than one generator function, consistent nilpotent connective systems can be obtained in a significantly different way with three naturally derived negation operators. As the class of non-strict t-norms has preferable properties that make them useful in constructing logical structures, the advantages of such systems are obvious [20]. Due to the fact that all continuous Archimedean (i.e. representable) nilpotent t-norms are isomorphic to the Łukasiewicz t-norm [17], the nilpotent systems studied earlier were all isomorphic to the well-known Łukasiewicz logic. Those consistent nilpotent connective systems which are not isomorphic to Łukasiewicz logic are called bounded systems (referring to the fact that the generators are bounded functions) [14]. Based on the results of [14] and [15], we now focus on equivalences in bounded systems.

The paper is organized as follows. After some preliminaries in Section 2, we define and examine the implication-based equivalences in bounded systems in Section 3. Next, we introduce and examine the so-called dual equivalences in Section 4. Using the arithmetic mean operator examined in Section 5, the aggregated equivalences are introduced and studied in Section 6. We show that unlike the other two types, the aggregated equivalences are threshold transitive and associative as well. In Section 7, for further applications in image processing, the overall equivalence of two grey level images is defined and an important semantic meaning of the aggregated equivalences is given. Finally, in Section 7, we summarize our key results.

2. Preliminaries

First we recall the basic notations and results regarding equivalences and nilpotent systems.

2.1. Equivalences

There exist several approaches to the definition of equivalences. Equivalences can be considered as binary relations [4,6–8,22,23].

Now we consider an equivalence as a connective. We give the definition of an equivalence as a binary operation on the unit interval according to Fodor and Roubens.

Definition 1. (See [16].) A function $e : [0, 1]^2 \to [0, 1]$ is called equivalence if it satisfies the following conditions:

1. Symmetry, i.e. $e(x, y) = e(y, x)$ for $\forall x, y \in [0, 1]$,
2. Compatibility, i.e. $e(0, 1) = e(1, 0) = 0$ and $e(0, 0) = e(1, 1) = 1$,
3. Reflexivity, i.e. $e(x, x) = 1$ for $\forall x \in [0, 1]$,
4. Monotonicity, i.e. $x \le x' \le y' \le y \Rightarrow e(x, y) \le e(x', y')$.