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A directory of families of infinitely extendible Archimedean copulas

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Abstract

This article introduces families of strict multivariate Archimedean copulas, C, the ordering of copulas within these families, and associated boundary copulas. The new families are compared with the Clayton, Frank, and Gumbel families with regard to measures of dependence, including tail dependence, Kendall's tau, and Spearman's rho for dimensions 2, 3, and 4. A new inequality that gives maximal values of $C(u, v) - uv$ is conjectured.

Keywords: Archimedean copula; Sklar's theorem; Kendall's tau; Spearman's rho; Joint distribution function

1. Introduction

A binary Archimedean copula is a copula C that can be represented in terms of a function f, strictly decreasing, convex, and continuous from $[0, \infty]$ onto $[0, 1]$, as follows:

$$C(u, v) = f(g(u) + g(v)), \tag{1}$$

where $g = f^{-1}$. Sklar's theorem in n dimensions (Nelsen [8], p. 46) entails joint distribution functions of the form

$$H(x_1, x_2, \ldots, x_n) = C(F_1(x_1), F_2(x_2), \ldots, F_n(x_n)),$$

a formulation which leads to defining a *multivariate Archimedean copula* as a function C that extends (1) for all $n \geqslant 2$:

$$C(u_1, u_2, \ldots, u_n) = f(g(u_1) + g(u_2) + \cdots + g(u_n)). \tag{2}$$

Nelsen ([8], p. 116–119) gives a convenient list of 22 families of copulas, of which nine ([8], p. 155) extend as in (2), including these three:

Clayton copulas: $g(u) = u^{-t} - 1$, $t > 0$
Frank copulas: $g(u) = -(\log(e^{-tu} - 1))/(e^{-t} - 1)$, $t > 0$
Gumbel (or Gumbel–Hougaard) copulas: $g(u) = (-\log u)^t$, $t \geqslant 1$

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To this list we add the product copula (also called the independence copula), $C(u,v)=uv$, generated by $g(u)=-\log u$ and denoted by Π. Another copula is given by $C(u,v)=uv/(u+v-uv)$, denoted in Nelsen [8] and below by $\Pi/(\Sigma-\Pi)$.

2. New families of copulas

Suppose that an Archimedean copula C is given by (1). A necessary and sufficient condition for f to be extendible as in (2) is that f be completely monotonic (Kimberling [6], Aslina, Frank, and Schweizer [1]). In order to access a large class of such functions f, recall that a function h is *absolutely monotonic* on (a,b) if for every nonnegative integer n, $h^{(n)}(u)$ exists and $h^{(n)}(u)\geq 0$ on (a,b), and a function f is *completely monotonic* on (a,b) if $(-1)^n f^{(n)}(u)\geq 0$ on (a,b). It is well known that if h is absolutely monotonic and f is completely monotonic, then $h\circ f$ is completely monotonic (e.g., Hofert [3], Lemma 1, part 4, which cites Widder [10] for a proof).

Suppose that h is absolutely monotonic on (a,b), and let $f(u)=h(1/u)$. It is easy to prove that f is completely monotonic on (a,b): first, $f'(u)=-(1/u^2)h'(1/u)\leq 0$, so that $-f'(u)\geq 0$ on (a,b); taking another derivative gives

$$f''(u)=2u^{-3}h'(1/u)+u^{-4}h''(1/u)\geq 0,$$

and the argument can be repeated to complete the proof by induction. A formula for $f^{(n)}(u)$ follows:

$$(-1)^n f^{(n)}(u)=\frac{T(n,1)h'(1/u)}{u^{n+1}}+\frac{T(n,2)h''(1/u)}{u^{n+2}}+\cdots+\frac{T(n,n)h^{(n)}(1/u)}{u^{2n}},$$
$$T(n,k)=(n-k)\binom{n-1}{k-1}\binom{n}{k},\quad k=0,1,\ldots,n.$$

The basic idea for generating certain new families of copulas can now be described as follows: start with an absolutely monotonic function $h(u)$, apply the transformation $u\to 1/u$, and adapt the result to obtain a completely monotonic function f from $[0,\infty]$ onto $[0,1]$. Families obtained in this manner are listed below. They were selected with two objectives in mind: (1) each f must have a closed-form inverse, and (2) each resulting copula should have some possible usefulness in dependence modeling, including extensions to hierarchical copulas (Mai and Scherer [7], Chapter 4), nested copulas (Hofert [4]), and vine copula models (Joe [5]).

Family C1: $f(u)=\sinh(\frac{t}{u+1})/\sinh t$,
$g(u)=-1+\frac{t}{\operatorname{arcsinh}(u\sinh t)}$, $t>0$

Family C2: $f(u)=[-1+\sec(\frac{t}{u+1})]/(-1+\sec t)$,
$g(u)=-1+\frac{t}{\operatorname{arcsec}[1+u(-1+\sec t)]}$, $0<t<\pi/2$

Family C3: $f(u)=\tan(\frac{t}{u+1})\cot t$,
$g(u)=-1+\frac{t}{\arctan(u\tan t)}$, $0<t<\pi/2$

Family C4: $f(u)=\frac{e^{t/(u+1)}-1}{e^t-1}$,
$g(u)=-1+\frac{t}{\log[1+u(e^t-1)]}$, $t>0$

Family C5: $f(u)=\arcsin(\frac{t}{u+1})/\arcsin t$,
$g(u)=-1+t\csc(u\arcsin t)$, $0<t\leq 1$