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Quotient MI-groups ☆

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Abstract

A many identities group (MI-group, for short) is a special algebraic structure in which identity like elements (called pseudoidentities) are specified and collected into a monoidal substructure. In this way, many algebraic structures, such as monoids of fuzzy intervals (numbers) or convex bodies possessing behavior very similar to that of a group structure, may be well described and investigated using a new approach, which seems to be superfluous for the classical structures. The concept of MI-groups was recently introduced by Holčapek and Štěpnička in the paper "MI-algebras: A new framework for arithmetics of (extensional) fuzzy numbers" to demonstrate how a standard structure can be generalized in terms of MI-algebras. This paper is a continuation of the development of MI-group theory and is focused on the construction of quotient MI-groups and a specification of the conditions under which the isomorphism theorems for groups are fulfilled for MI-groups.

Keywords: Many identities groups; MI-groups; Quotient MI-groups; Fuzzy numbers; Isomorphism theorems

1. Introduction

Data are typically collected using measurement procedures that can never provide fully precise and accurate results. Therefore, in data processing, some measurement uncertainty must be assumed, which requires techniques for handling vaguely specified quantities. Typical examples include computations with fuzzy or stochastic quantities, which express two basic phenomena present in the sample data – imprecision (incomplete information) and uncertainty. It is well known that treatments based on vaguely specified quantities can have an unpleasant consequence: some standard (arithmetic) rules that hold for numbers (crisp quantities) fail for non-crisp quantities. In this paper, we restrict our

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consideration to imprecisely specified quantities, where the imprecision is modeled using intervals or fuzzy intervals,[1] but the results can be adopted for use in other models (e.g., convex bodies [31–33]) or can be applied to stochastic quantities whose values are subject to random fluctuations.

It is well known in the literature (see, e.g., [4,3,8,16,18,23,26–28]) that the standard arithmetic of fuzzy quantities (numbers, intervals) based on Zadeh's extensional principle or the α-cut procedure does lack certain standard properties of the arithmetic of real numbers. In particular, the equalities

(i) $x + (-x) = 0$ and
(ii) $x \cdot x^{-1} = 1$

are not generally satisfied for a fuzzy quantity x, where 0 and 1 denote the crisp zero and the crisp unit element, respectively.[2] In other words, the problem lies in the non-existence of inverse elements for both arithmetic operations on fuzzy quantities. Note that the lack of inverse elements for fuzzy intervals has a convenient interpretation in terms of gradual numbers. Recall that a gradual number is a mappings of $(0, 1]$ into the set of real numbers; therefore, it express only fuzziness without imprecision (see, [6,7]). Then, a fuzzy interval is nothing more than a standard interval of gradual numbers. Hence, standard fuzzy arithmetic merely refers to operating with intervals of gradual numbers, where the inverse elements exist only for intervals of zero length. On the other side, gradual numbers do possess inverse elements, similarly to real numbers, and form a group structure.

To overcome the lack of inverse elements in the arithmetic of fuzzy intervals and cause the underlying algebraic structure to be a group or even a field, Klir [19] (see also [20]) has proposed a constrained fuzzy arithmetic, in which each operation is dependent on a requisite equality constraint that determines a rule defining how to proceed when a fuzzy interval occurs several times in a calculation. Independently of Klir's proposal, Lodwick [22] (see also [23,25,24]) has introduced the concept of constrained intervals, in which each interval $[\underline{x}, \overline{x}]$ is equivalently expressed as a linear function given by the triplet $(\underline{x}, \overline{x}, \lambda)$, where $\lambda \in [0, 1]$.[3] Postulating that the same intervals have the same lambda variables, Lodwick defined the arithmetic of constrained intervals in such a way that each arithmetic expression (e.g., $[\underline{x}, \overline{x}] - [\underline{x}, \overline{x}]$) is rewritten as the corresponding expression in terms of linear functions (e.g., $\underline{x} + \lambda(\overline{x} - \underline{x}) - \underline{x} - \lambda(\overline{x} - \underline{x})$); the minimum and maximum of this rewritten expression are then sought over all lambda variables λ_i that appear in the expression, which are subject to the constraints $0 \leq \lambda_i \leq 1$ (e.g., $[\underline{x}, \overline{x}] - [\underline{x}, \overline{x}] = \min/\max_{\lambda \in [0,1]}\{\underline{x} + \lambda(\overline{x} - \underline{x}) - \underline{x} - \lambda(\overline{x} - \underline{x})\} = [0, 0]$). In this way, similarly to Klir's constrained arithmetic, the existence of inverse elements is ensured and the corresponding algebraic structures over intervals become groups. Although constrained (fuzzy) interval arithmetic significantly improves the properties of basic operations with (fuzzy) intervals, the calculations based on constrained arithmetic can be very complex in some cases because, in general, the evaluation of constrained arithmetic expressions cannot be decomposed into a sequence of binary operations. For a discussion of how to improve the efficiency of constrained fuzzy arithmetic, we refer to [36].

In contrast to Klir's and Lodwick's ideas, which are based on a modification of standard (fuzzy) interval arithmetic, Mareš [27,26,28–30] regards the failure of basic arithmetic laws for fuzzy quantities as a natural consequence of the incomplete information contained in these quantities. This perspective has motivated him to introduce the aforementioned laws such that they hold up to an equivalence relation. In other words, he has proposed that for fuzzy quantities, the following statement holds:

$$x + (-x) = \tilde{0}, \qquad xx^{-1} = \tilde{1},$$

where $\tilde{0}$ and $\tilde{1}$ are symmetric fuzzy quantities with specific properties similar to those possessed by 0 and 1, respectively. From this point of view, Mareš naturally considers $\tilde{0}$ and 0 as well as $\tilde{1}$ and 1 to be "equivalent".

From the algebraic perspective, standard fuzzy interval arithmetic, for a broad class of various definitions of fuzzy intervals, leads to a commutative monoid (see, e.g., [5,29]). Bica [1] (see also [2]) was probably the first to characterize standard fuzzy arithmetic in a more specific way. As a suitable algebraic structure for fuzzy calculus, Bica proposed a commutative monoid in combination with a corresponding submonoid endowed with an involutive automorphism

[1] In this paper, we unify the terminology and use the notation "fuzzy interval" also in cases in which certain authors equivalently refer to fuzzy numbers (see, e.g., [7,6] for an explanation).

[2] Another problem that arises in fuzzy arithmetic is the failure of the distributivity law of the product for the sum (see, e.g., [5,29,31]).

[3] More precisely, the interval $[\underline{x}, \overline{x}]$ is redefined as a linear function of three parameters $(\underline{x}, \overline{x}, \lambda)$ such that $[\underline{x}, \overline{x}] \equiv \underline{x} + \lambda(\overline{x} - \underline{x})$, where $0 \leq \lambda \leq 1$.