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On approximations of Zadeh's extension principle

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Abstract

In this paper we provide several results concerning the approximation of Zadeh's extension of a given function. We provide several simple observations due to which we provide a general procedure allowing to approximate Zadeh's extension of any continuous map. Among other things, we demonstrated that the technique of F-transform can also be used to approximate this extension. Further, we studied how the choice of the metric on the space of fuzzy sets (resp. fuzzy numbers) can affect the quality of the approximation, and we clarified that, in comparison to recent habits, the choice of the levelwise metric need not be the most appropriate one, and that the approximation process could be simplified. Our results are general in the sense that we mostly require nothing but the continuity of the original map.

Keywords: Approximations; Endograph metric; Extension principle; F-transform; Levelwise metric; Endograph metric; Zadeh's extension

1. Introduction

The topic of this paper concerns one of the best known notions from fuzzy mathematics, namely a so-called Zadeh's extension (Zadeh's extension principle) introduced in the seventies. For some spaces X, Y and systems of fuzzy sets $\mathbb{F}(X), \mathbb{F}(Y)$ on them, Zadeh's extension represents a procedure which can be done for any map $f : X \to Y$ and which extends the map f to a map $z_f : \mathbb{F}(X) \to \mathbb{F}(Y)$ defined on the spaces of fuzzy sets on X by the formula

$$z_f(A)(x) = \sup_{y \in f^{-1}(x)} \{A(y)\} \tag{1}$$

for any $A \in \mathbb{F}(X)$. It is well-known that this concept is widely used in many constructions and applications.

It is also well-known that, in some practical applications, the calculation of Zadeh's extension z_f could really be a difficult task (e.g. [8]). The function f is known in general but usually the inverse f^{-1} is unknown and this makes a calculation of the set $f^{-1}(x)$ of preimages of a given point $x \in X$ difficult. Many authors already tried to cope with this problem and tried to find suitable approximations of Zadeh's extension. For instance, Kaufmann and Gupta in [17] presented a method based on interval arithmetics, some methods based on fuzzy weighted average were also

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presented in [31], Chalco-Cano and his colleagues in [2] presented a method based on the decomposition of fuzzy intervals and piece-wise linearization, and Stefanini et al. in [29] used some special representation of fuzzy numbers to solve this problem. For other references we refer the reader to the papers cited above.

It should be emphasized that in the papers above some specific assumptions were required and, as far as we know, no general solution to this problems is currently known. For instance, either some special kinds of fuzzy sets (e.g. LU fuzzy numbers), or decomposition of α-cuts and some kind of differentiability or monotonicity of the map f are required. Within this paper we would like to contribute to this problem and we would like to stress here that results contained in this work are general in the sense that we deal with ordinary continuous map and, mostly, with ordinary fuzzy sets.

The author's motivation to study Zadeh's extension comes from other branch of mathematics. The author has recently studied (e.g., [20,21]) some theoretical problems in the theory of fuzzy dynamical systems (fuzzy dynamical systems are given by Zadeh's extension z_f of a given crisp map $f : X \to X$) and realized that the problem of calculating of Zadeh's extension may complicate simulations in many practical applications. Moreover, it was an open question whether or not there exists a linguistic description of such fuzzy dynamical systems. If there would be such a linguistic representation, this would construct a bridge between fuzzy logic based methods on one side and analytic and topological methods on the other side, respectively. Note that there are also other mathematicians who recently tried (again under some specific assumptions) to find a good approximations of such fuzzy dynamical systems – see e.g. [2,29].

Within this paper we discuss approximations of Zadeh's extension from several points of view. This paper provides several basic and general observations due to which we are able to describe a general procedure allowing us to approximate Zadeh's extension of any continuous map on a compact metric space. We also demonstrate that the technique of F-transform is a suitable tool for approximation of Zadeh's extension of a given function. We discuss how the choice of a metric on the space of fuzzy sets can affect the efficiency of the approximation of Zadeh's extension (Section 3). It is shown that when the endograph metric is used the problem can be reduced to approximating only finitely many closed sets (Theorem 4). We also show that the problem of approximating map between spaces of fuzzy sets could be replaced by a problem of approximating a set-valued (resp. crisp (ordinary)) map. At the end of this paper we slightly discuss the possibility of using fuzzy rule based systems for this kind of approximation (see also [13] for distinct approach dealing with gradual rules). We would like to stress again that our results are general in the sense that we do not restrict our attention to special spaces of fuzzy sets and we require nothing but the continuity of the original map f. In the last but one section of this paper we specify a small class of maps which admits this behavior.

It should be noted that Zadeh's extension principle need not be used only to get fuzzy dynamical systems that are induced by maps $f : X \to X$. It is likely more common to use this principle in fuzzy arithmetics, where it is mainly used to create arithmetic operations between fuzzy numbers or fuzzy sets of various shapes. Probably one of the most important papers studying Zadeh's extension principle is [24] where also the problem of representation of fuzzy numbers via α-cuts is mentioned. After that, many other papers contributed to this research and many authors tried to find a good approximations of arithmetic operations between fuzzy numbers (e.g., [10]). Their attempts were similar to those that appeared in theory of fuzzy dynamical systems – some of those papers dealt with special representations of fuzzy numbers (e.g., [12,14,30]), some of them brought another methods (distinct from Zadeh's extension principle) to represent or to approximate those arithmetic operations (e.g., [9,12,15,25]). Nice surveys of most approaches can be found in [11,23]. Our results cannot be compared directly to those techniques, however we can say that our approach is different in the sense that it is sufficient to deal with finitely many α-cuts.

This paper is organized in an ordinary way. In Section 2 we recall some basic facts, then Section 3 contains the most important facts of this work. In the next section (Section 4) we provide the algorithm described above and in Section 5 we discuss some facts related to fuzzy rule based systems. Some concluding remarks are mentioned at the very end of this contribution in Section 6.