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ScienceDirect
Fuzzy Sets and Systems 283 (2016) 40–55
FUZZY sets and systems
www.elsevier.com/locate/fss

Necessary and sufficient conditions for the equality of the interactive and non-interactive sums of two fuzzy numbers

Lucian Coroianu

Department of Mathematics and Informatics, University of Oradea, 1 Universitatii Street, 410087 Oradea, Romania

Received 10 June 2013; received in revised form 28 October 2014; accepted 31 October 2014

Available online 6 November 2014

Abstract

This study aims to answer the open question proposed by Carlsson et al. (2004) [4], which asks whether it is possible to find the properties needed by a joint possibility distribution function such that the interactive sum with respect to this function and the standard sum of two fuzzy numbers will coincide. We provide the necessary and sufficient conditions such that this equality would hold. In addition, we find the necessary and sufficient conditions such that the interactive and standard sums will coincide at some point in the domain. In order to obtain an interactive addition that does not coincide with the standard addition, we show that we need to search for a joint possibility distribution that coincides as little as possible with the independent joint possibility distribution throughout the so-called diagonal of the two fuzzy numbers. As simple consequences, we deduce similar results for triangular norm-based additions of fuzzy numbers.

Keywords: Fuzzy number; Interactive sum; Joint possibility distribution

1. Introduction

The well-known extension principle [19] remains the basis of fuzzy arithmetic. This is the natural way to extend reasoning from real (crisp) values to fuzzy quantities. If $A_1, A_2, ..., A_n$ are fuzzy subsets of some set $\mathbb{X}$ and $f : \mathbb{X}^n \to \mathbb{R}$, then by the extension principle we obtain the fuzzy subset of $\mathbb{X}$, $f(A_1, ..., A_n)$, where for any $z \in \mathbb{X}$, we have $f(A_1, ..., A_n)(z) = \sup_{f(x_1,...,x_n)=z} \min\{A_i(x_i) : i = \overline{1, n}\}$. In the case of binary operations, we can express the fuzzy extension by using the strongest triangular norm $T_M(x, y) = \min\{x, y\}$ for all $x, y \in [0, 1]$ and hence we obtain

$$f(A, B)(z) = \sup_{f(x,y)=z} T_M\big(A(x), B(y)\big).$$

Therefore, by considering any triangular norm in the previous formula, we obtain a generalization of the extension principle. This approach has been studied extensively (e.g., see [6,7,11–17]), where this generalization was considered mainly with respect to the addition of fuzzy numbers (although an exception was [14], that considered the

E-mail address: lcoroianu@uoradea.ro.

http://dx.doi.org/10.1016/j.fss.2014.10.026

multiplication of fuzzy numbers) and triangular norm-based additions of fuzzy numbers (T-sums) were investigated. Triangular norm-based additions or multiplications have important advantages in practice, such as in shape-preserving problems (e.g., see [12,15,17]). In [8], the authors introduced the notion of a joint possibility distribution of fuzzy numbers $A_1, A_2, ..., A_n$ (see Definition 1 in Section 2). Basically, a joint possibility distribution of $A_1, ..., A_n$ is a function $C : \mathbb{R}^n \to \mathbb{R}$, for which the marginal distributions are represented by fuzzy numbers $A_1, ..., A_n$. Thus, we can say that the fuzzy numbers $A_1, ..., A_n$ interact through their possibility distribution C. Possibility distributions have been used successfully in statistical problems where possibility distributions are represented by fuzzy numbers instead of probability distributions (see [5,8,9]). Moreover, when C is a joint possibility distribution of fuzzy numbers $A_1, ..., A_n$, then an interactive extension principle is used, proposed by [4] (see Definition 2.1 in [4]). This approach is more general than the triangular norm-based extension principle (where the fuzzy numbers are fixed because the joint possibility distribution depends on the fuzzy numbers that represent the marginal distributions) because it is easy to prove that triangular norm-based extensions are particular cases of extensions through joint possibility distributions. The advantages of interactive extensions were described in the same study [4]. In general, for example, we know that we cannot have $A - A = 0$ unless A is a crisp value. However, the interactive subtraction of two fuzzy numbers with the same membership function can be 0 if their joint possibility distribution is selected such that they are completely correlated (see Definition 3.1 in [4]). The interactive additions are of particular interest. If C is a joint possibility distribution of fuzzy numbers A and B, then the interactive addition of A and B is denoted by $A +_C B$. The properties of interactive additions were studied by [2] and [4]. In [4], the following question was posed: “Let C be a joint possibility distribution with marginal distributions A and B. On what conditions will the equality $A +_C B = A + B$ hold?” In this case, $A +_C B$ is the interactive addition of fuzzy numbers A and B, while $A + B$ is the standard addition according to the extension principle. In the present study, we give the necessary and sufficient conditions such that this equality holds. Moreover, we show the same for the local problem of the equation $(A +_C B)(z) = (A + B)(z)$. The key element involved in the study of these problems is the so-called *diagonal* of two fuzzy numbers (see Definition 5 in Section 3). It seems that the values of C throughout the diagonal of A and B are essential for the study of the aforementioned problems. It should also be mentioned that the sufficient results cover the whole class of fuzzy numbers, while the necessary conditions hold for the class of fuzzy numbers with strictly monotone sides (we can also include the so-called one-sided fuzzy numbers, as they are referred to by [1]), which obviously is a class that is more than sufficient for applications. Since triangular norm-based sums are particular cases of interactive sums, it is easy to obtain the necessary and sufficient conditions for the equality of the triangular norm-based sum and the standard sum.

The remainder of this paper is organized as follows. In Section 2, we present some basic details of fuzzy numbers, joint possibility distributions, and triangular norms, as well as the interactive extension principle. Of independent importance, we also prove that we can always use the maximum instead of the supremum in the definition of the marginal distributions if the joint possibility distribution is upper semicontinuous in each variable, which is restricted to the support of the corresponding fuzzy number (Proposition 3). Furthermore, we recall the definition of the interactive extension principle (see Definition 4). Next, Section 3 presents the main results related to the study of the equality $A +_C B = A + B$ and the equation $(A +_C B)(z) = (A + B)(z)$. Several characterizations are provided. In Theorems 10–11, we obtain the necessary and sufficient conditions for the equality $(A +_C B)(z) = (A + B)(z)$ for some $z \in \mathbb{R}$. In Theorem 12 and Corollaries 14–15, we obtain the necessary and sufficient conditions for the equality $A +_C B = A + B$. As mentioned earlier, we need to investigate the values of C throughout the diagonal of A and B. In particular, we need to compare the values of C with the values of C_{id}, where C_{id} represents the independent joint possibility distribution, i.e., $C_{id}(x, y) = A(x) \wedge B(y)$ for all $(x, y) \in \mathbb{R}^2$. This section also provides several examples and finally, for fuzzy numbers with strictly monotone sides, the weakest upper semicontinuous joint possibility distribution is found such that the interactive sum agrees with the standard sum. In Section 4, we discuss the particular case of T-sums. We obtain the results easily because they are simple consequences of those given in Section 3. We should note that in the case where the fuzzy numbers are continuous with strictly monotone sides, then the strongest triangular norm T_M is the unique upper semicontinuous triangular norm for which the T-sum agrees with the standard sum (see Theorem 19). Finally, we give our conclusions and summarize the main results, as well as suggesting further research in this area.