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Subsethood measures for interval-valued fuzzy sets based on the aggregation of interval fuzzy implications

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Abstract

The connection between fuzzy subsethood measures and fuzzy implications straightforwardly follows from the definition of inclusion for (crisp) sets. In accordance with this connection we introduce a new constructive method for (weak) fuzzy subsethood measures for interval-valued fuzzy sets based on the aggregation of fuzzy interval implications. For this purpose we generalize aggregation functions and propose a technique for aggregation of intervals. The benefit of our approach lies in the wide range of obtained fuzzy subsethood measures for interval-valued fuzzy sets. Moreover, the method gives the possibility to construct new fuzzy subsethood measures with properties that are appropriate to the situation under investigation. Another advantage of the method is its clear semantic interpretation.

Keywords: Subsethood measure; Interval valued fuzzy sets; Interval aggregation; Aggregation function; Fuzzy implication; Inclusion indicator; Lipschitz; Best interval representation

1. Introduction

An inclusion for fuzzy sets A, B in a set U was defined by Zadeh [1] as follows

$$A \subseteq B \qquad \text{iff} \qquad (A(x) \leq B(x), \quad \forall x \in U). \tag{1}$$

This is a binary relation, a fuzzy set A is completely contained within B or it is not (A is subset or is not subset of B). In fuzzy logic, it is more natural to consider an indicator of degree to which A is subset of B. In general, such an indicator, called a fuzzy *subsethood measure* (or *inclusion indicator*), is a mapping $\sigma : FS(U) \times FS(U) \to [0, 1]$, where $FS(U)$ denotes the class of fuzzy sets in a universe U. Pioneers of inclusion indicators are Bandler and Kohout [2]; since then subsethood measures have been considered by many researchers, e.g. [3–7].

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One of the most important generalizations of fuzzy sets (FSs, in abbreviation) are interval-valued fuzzy sets (IVFSs) introduced by Zadeh [8]. There are two different visions of IVFSs. They are fuzzy sets whose membership grades are intervals; or they are fuzzy sets whose membership grades are precise values but our incomplete knowledge of these values is represented by intervals. See the discussion in [9]. The foundations of this paper follow from the second vision.

The authors of [10] propose axiomatic requirements a measure of subsethood for IVFSs should comply with, and some similar approach is used in [11] and [12]. In these papers, the subsethood measure for IVFSs is defined as a mapping $\sigma : IVFS(U) \times IVFS(U) \to [0, 1]$, where $IVFS(U)$ denotes the class of IVFSs in a universe U, i.e., the subsethood is only one number from $[0, 1]$. Bustince [13] investigates in detail axioms for subsethood measures for interval-valued fuzzy sets, and concludes that the measure should be rather interval, i.e., the subsethood measure should be a mapping $\sigma : IVFS(U) \times IVFS(U) \to \mathbb{D}$, where $\mathbb{D}$ denotes the set of closed subintervals of $[0, 1]$. Our approach is based on this conclusion; moreover, for reasons stated in [7], we adjust the axioms of Young [14] (which were proposed for FSs) to IVFSs. A different view of the issue is presented in [15], where the proposed subsethood measure takes values in a Boolean lattice, in other words, the subsethood is viewed as an L-fuzzy valued relation between fuzzy sets.

Subsethood measures of fuzzy sets have been used in many applications, e.g. approximate reasoning [13], classification [16], computing with words [17] and control [18]. Recently many fuzzy sets applications are generalized to interval-valued fuzzy sets (or interval type-2 fuzzy sets or Atanassov's intuitionistic fuzzy sets). Hence it is important to deeply study theoretical aspects of subsethood measures in the settings of IVFSs and to propose some appropriate formulas for the measures. Moreover, our results can be easily adapted to Atanassov's intuitionistic fuzzy sets settings (see [19,20] for relations between the two generalizations of fuzzy sets).

In this paper, we propose a novel constructive method for fuzzy subsethood measures for IVFSs. We apply so-called best interval representation, which is a new approach to subsethood measures, and a similar approach can be used to fuzzy entropy, distance measure and similarity measure (that are three basic concepts of fuzzy sets theory [4]). Our method was inspired by work of Bustince et al. [21] who defined so-called DI-subsethood measures constructed as an aggregation of fuzzy implications. We adapt their work to IVFSs. For this purpose we generalize aggregation functions and propose a technique for aggregation of intervals. Then we introduce a new constructive method for fuzzy subsethood measures for IVFSs based on the interval aggregation of fuzzy interval implications.

Note that our approach takes into account the specific nature of intervals: the resulting subsethood measure of IVFSs $\widetilde{A}, \widetilde{B}$ computed via our method is an interval whose midpoint may not be equal to the DI-subsethood measure of FSs A, B given by midpoints of intervals representing membership grades of $\widetilde{A}, \widetilde{B}$.

The connection between fuzzy subsethood measures and fuzzy implications is clear and very intuitive. It follows from the definition of inclusion for (crisp) sets A, B:

$$A \subseteq B \qquad \text{iff} \qquad (\forall x)(x \in A \to x \in B). \tag{2}$$

So our method has clear semantic interpretation and our approach is in accordance with an intuitive understanding of subsethood measures. This gives reasons to believe that it will be appropriate also from the point of view of some real-world applications. Moreover, the method gives the possibility to construct new fuzzy subsethood measures with properties that are appropriate to the situation under investigation. The properties of obtained fuzzy subsethood measure depend on the properties of chosen fuzzy implication and aggregation function which are well-known.

The paper is organized as follows. Section 2 contains basic definitions and notations that are used in the remaining parts of the paper. Section 3 presents the notion of interval representation of a real function. Section 4 describes interval fuzzy negations and Section 5 interval fuzzy implications. Section 6 introduces interval aggregation functions. Section 7 discusses the notions of fuzzy subsethood measures for IVFSs. In Section 8, we propose a new constructive method for fuzzy subsethood measures for IVFSs based on the interval aggregation of fuzzy interval implications. Some examples are given in Section 9; and finally, in Section 10 we show that our approach takes into account the specific nature of intervals. The conclusions are discussed in Section 11.

2. Preliminaries

In this section we recall some well-known concepts that will be useful for subsequent developments in the paper.