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Necessary and sufficient conditions for generalized uniform fuzzy partitions ☆

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Abstract

The fundamental concept in the theory of fuzzy transform (F-transform) is the fuzzy partition, which is a generalization of the classical concept of partition. The original definition assumes that every two normal fuzzy subsets in a partition overlap in such a way that the sum of the membership degrees at each point is equal to 1. This condition can be generalized by relaxing the assumption of normality for fuzzy sets. The result is a denser fuzzy partition that may improve the approximation properties and the smoothness of the inverse F-transform. A fuzzy partition with this property will be referred to as *general*. The problem is how a general fuzzy partition can be effectively constructed. If we use a generating function with special properties, then it is not immediately clear whether it defines a general fuzzy partition. In this paper, we find a necessary and sufficient condition that will enable the optimal generalized fuzzy partition to be designed more easily, which is important in various practical applications of the F-transform, for example, image processing, time series analysis, and solving differential equations with boundary conditions.

Keywords: Fuzzy transform; Uniform fuzzy partition; F-transform

1. Introduction

According to the classical definition, the partition of a given set X consists of a system of sets $\{A_i \subset X \mid i \in I\}$ such that $\bigcup\{A_i \mid i \in I\} = X$ (covering) and $A_i \cap A_j = \emptyset$ (disjointness) holds for all $i \neq j$.[1] The idea to replace sets

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[1] The set I is a set of subscripts that can be finite or infinite.

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A_i with fuzzy sets on X was probably first suggested by Ruspini in paper [41]. The result is called a *fuzzy partition* of X provided that the sum $A_i(x) + A_{i+1}(x) = 1$ holds for all $i \in I$ and $x \in X$. In this paper, we will call this condition the *Ruspini condition*. A fuzzy partition of X is called *uniform* if X is linearly ordered (we assume $X = \mathbb{R}$) and all fuzzy sets A_i are regularly shifted copies of one initial fuzzy set, and therefore, this set is called the generating fuzzy set. The main advantage of this concept is that there are no crisp borders between (fuzzy) sets A_i. Ruspini's idea initiated deep research in the field of fuzzy partitions. In the literature, various definitions of fuzzy partitions have been proposed. Most of them generalize classical axioms, for example, the covering and disjointness properties (cf., [5,6,9,14,21,25,27,31]). The need to find a one-to-one correspondence between fuzzy partitions and fuzzy equivalence relations (which is analogous to the classical correspondence between equivalences and equivalence classes) motivated researchers to establish a more appropriate type of fuzzy partition so that some difficulties encountered in the previous definitions (see, e.g., [4,15,20,24,26,48]) could be overcome.

The usefulness of fuzzy partitions was recognized in various fields of applications of fuzzy set theory, for example, in fuzzy pattern recognition [2,3,55], fuzzy control [16,28], fuzzy relation equations [38,50], fuzzy histogram estimation [23,47,54] and elsewhere. In this paper, we focus on the notion of a fuzzy partition, which is a background notion in *fuzzy transform* [33,34,49]. However, our result can be used in other applications, for example, in fuzzy control or fuzzy density estimation.

Recall that the fuzzy transform (F-transform) is a special soft computing technique originally proposed by Perfilieva in [33] (see also [35]) that has many applications in various fields, for example, in data analysis [11,13,39], image processing [7,10,12,34], approximate solutions of differential equations [37,52,53], time series analysis [29,30,40,51], probability density estimation [17], and non-parametric regression [18,19].

The core of the F-transform technique consists in transforming a continuous (i.e., infinite) function f into a finite vector of numbers and then transforming it back. Of course, the inverse F-transform gives only an approximation of the original function. However, by setting the parameters properly we can obtain an approximation with the desired properties.

More precisely, the F-transform starts by introducing a fuzzy partition of a given interval of reals using fuzzy sets with a special shape; in the theory of F-transform, these sets are called *basic functions*. The F-transform has two phases: direct and inverse. Whereas the *direct* F-transform transforms a bounded real function f into a finite vector of real numbers based on the given fuzzy partition, the *inverse* F-transform returns this finite vector to an approximation of the original function. Thus, the result of the inverse F-transform is a function $\hat{f}$ that approximates f.

The F-transform can be applied to functions of many variables (see [8,11,13]) using a multidimensional fuzzy partition that is obtained from one-dimensional ones using products of basic functions.[2] It should be noted that the F-transform technique has *very low computational complexity*.[3]

The original definition of the F-transform deals with specifically shaped basic functions that make fuzzy partitions of real intervals, where each two neighboring basic functions overlap and satisfy the Ruspini condition. In addition to the important result in [35, Corollary 2], which states that under certain conditions a sequence of inverse F-transforms $\hat{f}$ uniformly converges to the original function f, several authors (see [18,22,43,44]) focused on the problem of the smoothness of $\hat{f}$. It was shown that better smoothness of the approximating function $\hat{f}$ can be obtained if we consider a denser fuzzy partition in which more than two basic functions can overlap. We speak about the *generalized fuzzy partition*. This idea was originally suggested in [43] and further investigated in [1,18,22,44]. A theoretical investigation of higher smoothness of the inverse F-transform for generalized fuzzy partitions using the concept of total variation[4] was provided in [18,43,44].

The difference in smoothness is illustrated in Fig. 1, where the estimation of a trend-cycle of a time series using the inverse F-transform based on two types of fuzzy partition is depicted. Note that we used the same shape basic functions in both cases of fuzzy partitions, and the smoother trend-cycle estimation depicted on the right side of the figure was achieved by overlapping more than two basic functions.[5]

[2] This simple method for extending one-dimensional basic functions to multidimensional ones can also be found, for example, in kernel smoothing theory, where multivariate kernels are defined by products of univariate kernels (see [32,42]).

[3] For an example, please consult the comparison of Nadaraya–Watson and F-transform-based estimators presented in [18].

[4] For the definition of total variation, we refer the reader to [56].

[5] For example, a higher number of overlapping basic functions in a fuzzy partition may provide a more accurate derivation of the buy/sell signal (see [46]).