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A Stone-type duality for sT_0 stratified Alexandrov L-topological spaces

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Abstract

We show that for every complete lattice A, both the set of completely prime elements and the set of completely coprime elements are one-to-one corresponding to $\mathrm{CH}(A)$, the set of complete lattice homomorphisms from A to the two-element lattice $\mathbf{2}$. A complete lattice is called completely generated, a cg-lattice for short, if it is generated by the set of completely prime elements, or equivalently, by the set of completely coprime elements. Then we restudy the duality between the category of T_0 Alexandrov topological spaces and the category of cg-lattices by means of $\mathrm{CH}(A)$. With these preparations, for a frame L as the truth value table, we introduce sT_0 separation axiom for stratified Alexandrov L-topological spaces, and finally establish a duality between the category of sT_0 stratified Alexandrov L-topological spaces and the category of completely generated complete L-ordered sets. We also investigate some properties of the sT_0 axiom, for example, it is hereditary by closed subspaces and productive.

Keywords: Topology; Category theory; Duality; Complete (co)prime element; Complete lattice homomorphism; Completely generated lattice; (Complete) L-ordered set; Alexandrov L-topology; sT_0 space

1. Introduction

In mathematics, a topological duality refers to a kind of categorical dualities between certain categories of topological spaces and categories of partially ordered sets. Today, these dualities are usually collected under the name *Stone duality*, since they form a natural generalization of the famous Stone Representation Theorem for Boolean algebras [34]. Stone-type dualities also provide the foundation for pointfree topology and are exploited in theoretical computer science for the study of formal semantics [1,17].

The famous Papert–Papert–Isbell adjunction [14,26] between topological spaces and locales provides an appropriate environment in which to develop both point-set topology and the theory of locales (pointfree topology). This

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adjunction also gives rise to the concept of sobriety, which plays an important role in several Stone Representation Theorems [17].

But there are a lot of interesting spaces which are not sober. For example posets taken with the Alexandrov topology are not always sober [17]. Domains, that is continuous dcpos,[1] equipped with the Scott topology are sober spaces [9], but Johnstone [16] showed that not every dcpo with the Scott topology was sober. Therefore, it is interesting in both mathematics and theoretical computer science to study Stone duality for spaces which need not be sober. For example in [6], Bonsangue, Jacobs and Kok defined and studied observation frames and used them to establish dualities for T_0 spaces, T_1 spaces, open compact spaces, core compact spaces and T_0 Alexandrov topological spaces, etc.

Precisely in [6], it is shown that the duality for T_0 Alexandrov topological spaces are those complete lattices order-generated by completely prime elements. Since for a frame A, the set of frame homomorphisms from A to the two-element lattice $\mathbf{2} = \{0, 1\}$ is bijective to the set of prime elements or completely prime filters of A [17], we could guess that for a complete lattice A, the completely coprime elements are closely related to some special maps from A to $\mathbf{2}$. Explicitly, for a complete lattice A, we would like to replace completely prime elements of A by certain special homomorphisms from A to $\mathbf{2}$ and then restudy the duality between T_0 Alexandrov topological spaces and those complete lattices mentioned above. Such a restudy can help us to generalize the classical results into lattice-valued setting. For example, in locale theory in lattice-valued setting, by using L-frame homomorphisms instead of certain fuzzy version of prime elements, Yao [38] established a duality between spatial L-frames (defined by L-orders) and modified L-sober spaces, for L a fixed frame as the truth value table.

In this paper, we try to generalize the duality between T_0 Alexandrov topological spaces and those complete lattices order-generated by completely prime elements [6] into lattice-valued setting. Firstly, we show that for every complete lattice A, both the set of completely prime elements and the set of completely coprime elements are one-to-one corresponding to $\mathrm{CH}(A)$, the set of complete lattice homomorphisms from A to $\mathbf{2}$. We show that for every complete lattice, it is order-generated by the set of completely prime elements if and only if it is order-generated by the set of completely coprime elements. Hence, we have reasons to call such a complete lattice a completely generated lattice, a cg-lattice for short. Then we restudy the duality between the category of T_0 Alexandrov topological spaces and the category of cg-lattices by replacing the set of completely prime elements with $\mathrm{CH}(A)$. For lattice-valued setting with a fixed frame L as the truth value table, we construct an adjunction between the category of stratified Alexandrov L-topological spaces and the category of complete L-ordered sets. This adjunction induces a duality between the category of sT_0 stratified Alexandrov L-topological spaces and the category of completely generated complete L-ordered sets, as desired.

2. Preliminaries

In this section, we will recall some basic concepts and results on categories, lattices, (fuzzy) topology and L-ordered sets, which will be used through out this paper.

2.1. Category theory

For category theory, we refer to [3].

For two objects A and B in a category $\mathbf{C}$, we would like to use $[A, B]_{\mathbf{C}}$ to denote the set of $\mathbf{C}$-morphism from A to B, by $|\mathbf{C}|$ the class of $\mathbf{C}$-objects and by $Mor(\mathbf{C})$ the class of $\mathbf{C}$-morphisms.

Let $F : \mathbf{A} \longrightarrow \mathbf{B}$ be a functor and $B \in |\mathbf{B}|$. A pair (u, A) with $A \in |\mathbf{A}|$ and $u : B \longrightarrow F(A) \in Mor(\mathbf{B})$ is called *universal* for B with respect to F, provided that for every $A' \in |\mathbf{A}|$ and every $\mathbf{B}$-morphism $f : B \longrightarrow F(A')$, there exists a unique $\mathbf{A}$-morphism $\overline{f} : A \longrightarrow A'$ such that $F(\overline{f}) \circ u = f$. Dually, a pair (A, u) with $A \in |\mathbf{A}|$ and $u : F(A) \longrightarrow B \in Mor(\mathbf{B})$ is called *co-universal* for B with respect to F, provided that for every $A' \in |\mathbf{A}|$ and every $\mathbf{B}$-morphism $f : F(A') \longrightarrow B$ there exists a unique $\mathbf{A}$-morphism $\overline{f} : A' \longrightarrow A$ such that $u \circ F(\overline{f}) = f$.

Let $F : \mathbf{A} \longrightarrow \mathbf{B}$ and $G : \mathbf{B} \longrightarrow \mathbf{A}$ be two functors. F is called a *left adjoint* of G (or G a *right adjoint* of F) or (F, G) is an *adjunction* between $\mathbf{A}$ and $\mathbf{B}$, in symbols $F \dashv G : \mathbf{A} \rightharpoonup \mathbf{B}$, if for every $A \in |\mathbf{A}|$, there exists a universal pair $(u_A, F(A))$ with respect to G, or equivalently, for every $B \in |\mathbf{B}|$, there exists a co-universal pair $(G(B), u_B)$ with respect to F.

[1] A poset is called a directed-complete poset [9] (a dcpo, for short) if every directed subset has a join.