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On the lattices of L-topologies

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Abstract

Certain properties of the lattices of L-topologies determined by the families of Scott continuous functions for a given topological space are investigated. The authors disprove certain known theorems on the above lattices and the correct results are provided using the concept of strict chain join-generability in lattices.

Keywords: Topology; Scott continuous function; Strict chain join-generable lattice; Atom; Dual atom

1. Introduction

Lattice theory and topology are two related branches of mathematics, each influencing the other. Many authors have already undertaken the study of the lattice structure of the set of all topologies on a given set [1,2]. It has been proved that this lattice is complete, atomic, dually atomic and complemented but neither modular nor distributive. Analogously, the lattice structure of the sets of L-topologies on a given set has interested many authors [3–6]. In a similar manner, the lattice structure of the families of weakly induced L-topologies and that of stratified L-topologies have been investigated in [7,8].

In [4,7,8] the authors have examined the lattices $S_{\tau,L}$ (consisting of the collection of L-topologies determined by the families of Scott continuous functions from a topological space (X,τ) to L), W_τ (consisting of the collection of weakly induced L-topologies determined by the family of Scott continuous functions from a topological space (X,τ) to L) and $L(X)$ (the collection of all stratified L-topologies on X) and have obtained some of their properties regarding the existence of atoms, dual atoms and complements of atoms.

However, in this paper we give counter examples for some of the claimed properties of these lattices. In pursuit of the correct versions of these results, we have initiated a study on the structure of strict chain join-generable (*scjg*)

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lattices and have derived their certain properties. It has been proved that a completely distributive lattice is *scjg* if and only if it has no adjacent elements.

Further, we establish that the claimed properties of $S_{\tau,L}$, W_τ and $L(X)$ can stand with the help of strict chain join-generability of L. The notion of a strongly *scjg* lattice is introduced and is proved that $S_{\tau,L}$ has no dual atom, if the lattice L is strongly *scjg*. Though the converse is not true in general, it holds in a chain if (X,τ) admits the smallest non-empty closed set. Again, $L(X)$ has no atom if L is *scjg* and the converse holds, if L is a chain. Further, in the case of chains, strict chain join-generability is necessary and sufficient to rule out complements for atoms of the form $\{\underline{0}, \underline{1}, f\}$; support $f = X$ and of the form $\{\underline{0}, \underline{1}, \underline{\alpha}\}$ in lattices $S_{\tau,L}$ and W_τ respectively.

2. Preliminaries

Throughout this paper, X stands for a non-empty ordinary set and L for a lattice with the smallest element 0 and the greatest element 1. The constant function in L^X, taking the value α is denoted by $\underline{\alpha}$. The characteristic function of $A \subseteq X$ is denoted by μ_A. The set of all natural numbers is denoted by $\mathbb{N}$ and $\{1, 2, 3, \cdots, n\}$ is denoted by $\mathbb{N}_n$. The fundamental definitions of L-fuzzy set theory and L-topology are assumed to be familiar to the readers.

Definition 2.1. (See [10].) An element $p \neq 1$ of L is called prime if $a \wedge b \leq p$, then $a \leq p$ or $b \leq p$ for $a, b \in L$. The set of all prime elements of L is denoted by $pr(L)$.

Definition 2.2. (See [10].) An element of L is called an atom if it is a minimal element of $L \setminus \{0\}$.

Definition 2.3. (See [10].) An element of L is called a dual atom if it is a maximal element of $L \setminus \{1\}$.

Definition 2.4. (See [10].) An element α in L is called join-irreducible if $\alpha > 0$ and for all $a, b \in L$, $\alpha = a \vee b \Longrightarrow \alpha = a$ or $\alpha = b$.

Notation. (See [10].) Let P be a poset and $a \in P$, $\uparrow a = \{b \in P : b \geq a\}$.

The following pentagon (Fig. 1) and diamond (Fig. 2) lattices are commonly denoted by N_5 and M_3 respectively.

Theorem 2.5. *(See [9].)* (i) *L is non-modular if and only if $N_5 \rightarrowtail L$.*

(ii) *L is non-distributive if and only if $N_5 \rightarrowtail L$ or $M_3 \rightarrowtail L$, where $N \rightarrowtail L$ means that L contains a sublattice isomorphic to N.*

Definition 2.6. (See [10].) A completely distributive lattice L is called an F-lattice if it has an order reversing involution $' : L \longrightarrow L$.

Definition 2.7. (See [11].) The Scott topology on an F-lattice L is the topology generated by the sets of the form $\{t \in L : t \nleq p\}$ where $p \in pr(L)$.

Definition 2.8. (See [11].) Let (X, τ) be a topological space. A continuous function $f : (X, \tau) \to L$ where L has its Scott topology is said to be a Scott continuous function.

Definition 2.9. (See [10].) Let (X, τ) be a topological space. An L-topological space (X, δ) is said to be weakly induced if each element of δ is Scott continuous from $(X, [\delta])$ to L, where $[\delta] = \{A : \mu_A \in \delta\}$.

Definition 2.10. (See [10].) An L-topology on X is said to be stratified if it contains all constant L-subsets $\underline{\alpha}$.

Definition 2.11. (See [11].) Let (X, τ) be a topological space. The L-topology $\omega_L(\tau) = \{f \in L^X \mid f : (X, \tau) \to L$ is Scott continuous$\}$ is called the induced L-topology.