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Fuzzifying ideal convergence in fuzzifying topological linear spaces ☆

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Abstract

In this study, we propose a framework for fuzzifying ideal convergence in fuzzifying topological linear spaces. After defining fuzzifying ideal convergence, we discuss many properties of fuzzifying ideal convergence in fuzzifying topological linear spaces. We also introduce a stronger convergence than fuzzifying ideal convergence, and a sufficient condition (called (AP) fuzzifying condition) under which fuzzifying ideal convergence implies the stronger convergence above is proved.

Keywords: Fuzzifying ideal; Fuzzifying ideal convergence; Fuzzifying ideal* convergence; Fuzzifying topological linear space; Topology

1. Introduction

The study of convergence theory is an important research topic in classical mathematics. As a generalization of classical convergence, Fast [1] introduced the notion of statistical convergence of sequences of real numbers. Subsequently, many studies have investigated the properties of statistical convergence. At present, statistical convergence has several applications in different fields of mathematics, including number theory [2], summability theory [3], probability theory [4], topology [5], and fuzzy analysis [6,7]. Kostyrko and Šalát [8] generalized the notion of statistical convergence to the concept of ideal convergence (i.e., $\mathcal{I}$-convergence) of sequences in metric spaces and they established the fundamental framework of $\mathcal{I}$-convergence. The study of $\mathcal{I}$-convergence has been investigated widely [9–13].

The idea of the fuzzification of openness was initiated by Höhle [14] and subsequently extended to fuzzy topology by Šostak [15] in 1985. Using the semantic method of continuous-valued logic, Ying [16–18] established a fundamental framework for fuzzifying topology. The theory of classical topological linear spaces is an important foundation of modern analysis, so Qiu [19] introduced the notion of fuzzifying topological linear spaces by giving a characteristic theorem of fuzzifying neighborhood bases of zero element in fuzzifying topological linear spaces, as well as obtaining

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some properties of the T_2 separation axiom for fuzzifying topological linear spaces. Based on the framework given by Qiu, Yan [20] studied some of the properties of continuous linear operators from one fuzzifying topological linear space to another, where the well-known closed graph of linear operators in classical topological linear spaces was generalized to fuzzifying topological linear spaces.

The theory of sequences convergence plays an important role in study about the property of boundedness in fuzzifying topological linear spaces. Especially, in the research of locally bounded fuzzifying topological linear spaces and the sequential compactness of a set (see [19,20]). As a generalization of sequences convergence, we hope that the notion fuzzifying ideal convergence brings more general results on boundedness and compactness in the theory of fuzzifying topological linear spaces. The main aim of the present study is to propose a fundamental framework for the theory of fuzzifying ideal convergence in fuzzifying topological linear spaces. Indeed, based on Šostak, Yue and Huang [21] introduced the notion of a fuzzifying ideal in their study of topological molecule lattices. Our approach is slightly different from that proposed by Yue and Huang because we introduce the concept of a fuzzifying ideal in the set of natural numbers by using the semantic method of Łukasiewicz logic. The reason is we only discuss fuzzifying ideal convergence of sequences here, and the domain of sequences is just as the set of natural numbers. We also introduce the notions of fuzzifying ideal convergence and fuzzifying ideal* convergence, as well as discussing some basic properties of these types of convergence. In fact, we will find fuzzifying ideal* convergence is stronger than fuzzifying ideal convergence, and a relation between the subsequences convergence and fuzzifying ideal convergence will be proved by means of fuzzifying ideal* convergence. Finally, we obtain as a conclusion that fuzzifying ideal convergence implies fuzzifying ideal* convergence with the help of the fuzzifying condition (AP). Moreover, considering the relation between the roughness of a set (see [22]) and fuzzifying ideal in incomplete information systems, these discussions may be beneficial for studies of fuzzy information systems.

2. Preliminaries

In this section, we explain some concepts and results related to fuzzifying topology and a fuzzifying topological linear space, which we use in the sequel. For further details, please refer to [16–20]. First, we define a number of fuzzy logical notations and properties.

For any formula φ, the symbol $[\varphi]$ denotes the truth value of φ, where the set of truth values is the unit interval $[0, 1]$. A formula φ is valid and we write $\models \varphi$ if and only if $[\varphi] = 1$ for every interpretation.

$$(\alpha \to \beta) \overset{\text{def}}{=} \min(1, 1 - [\alpha] + [\beta]);$$

$$[\alpha \wedge \beta] \overset{\text{def}}{=} \min([\alpha], [\beta]);$$

$$\alpha \bigvee \beta \overset{\text{def}}{=} \neg(\neg\alpha \bigwedge \neg\beta).$$

Throughout this study, X always denotes a universe of discourse. According to the terminology introduced by Rodabaugh [23], the notation $f^{\to}$ denotes a special universal lifting of the original function f, and $f^{\leftarrow}$ denotes the unique right adjoint of the image operator f^{-1} of the original function guaranteed by the Adjoint Functor Theorem. For any set X and $x \in X$, $\dot{x}$ denotes the principal filter generated by x, i.e., $\dot{x} = \{U \in 2^X \mid x \in U\}$.

Definition 2.1. (See Šostak [15], Ying [16].) For each non-empty set X and every mapping $\tau : 2^X \to I$, the pair (X, τ) is called a fuzzifying topological space if and only if τ fulfills the following conditions.

(1) $\models X \in \tau$,
(2) $\models (\forall A_1,\ A_2 \in 2^X)(A_1 \in \tau \bigwedge A_2 \in \tau \to A_1 \cap A_2 \in \tau)$,
(3) $\models (\forall \mathscr{A})(((\mathscr{A} \subseteq 2^X) \bigwedge (\forall A \in \mathscr{A})(A \in \tau)) \to \bigcup_{A \in \mathscr{A}} A \in \tau)$.

Definition 2.2. (See Ying [16].) Let (X, τ) be a fuzzifying topological space, $x \in X$. The fuzzifying neighborhood system of x is denoted by $\mathscr{N}_x : 2^X \to [0, 1]$ and defined as

$$A \in \mathscr{N}_x :\Leftrightarrow\ \exists B((B \in \tau) \wedge (x \in B \subseteq A)).$$

Clearly, for each $A \in 2^X$, $\mathscr{N}_x(A) = \bigvee_{x \in B \subseteq A} \tau(A)$. By [16, Theorem 3.2], the mapping of $\mathscr{N}_x$ has the following properties: for all $U, V \subseteq X$,