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Closedness of the category of liminf complete fuzzy orders

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Abstract

It is shown that the category of liminf complete fuzzy orders and liminf continuous maps valued in the complete residuated lattice based on a continuous t-norm on the unit interval is monoidal closed; and that this category is Cartesian closed if and only if the t-norm is the minimum.

Keywords: Category theory; t-norm; Fuzzy order; Liminf completeness; Monoidal closed category; Cartesian closed category

1. Introduction

In 1971, Zadeh introduced the notion of fuzzy preorders [29]. This notion was extended later to the setting that the truth-value table is the unit interval [0, 1] equipped with a left continuous t-norm, and more generally, a complete residuated lattice $\mathcal{Q} = (Q, \&)$ [2,14,17].

Cartesian closedness, and monoidal closedness in general, is a property of fundamental importance in category theory [19,20,22]. We remind the reader of the fact that a complete residuated lattice is itself a monoidal closed category. The Cartesian closedness of the category **DCPO** of directed complete partially ordered sets and Scott continuous maps is the main reason for its prominent role in domain theory, as explained in [9]: "One of the most noteworthy features of **DCPO** is that it is Cartesian closed. Not only is this fundamental for the application of continuous lattices and domains to logic and computing, but it also provides evidence of the mathematical naturalness of the notion." Because of this, there have been continuous efforts to search for quantitative versions of directed completeness and Cartesian closed categories of fuzzy orders (partially ordered sets valued in a complete residuated lattice) [10,21,27]. The situation is much more complicated in the fuzzy setting. First, there exist different approaches to a quantitative notion of directed completeness because of the lack of a "standard" one [3,6–8,16,18,23–26,28]. Second, unlike its classical counterpart, the category **Ord**($\mathcal{Q}$) of $\mathcal{Q}$-ordered sets and order preserving maps is not Cartesian closed in general [4,5], thus, the requirement for a subcategory of fuzzy orders to be Cartesian closed seems to be too demanding. However, the category **Ord**($\mathcal{Q}$) is always monoidal closed [20,25,26], so, it is natural to search for monoidal

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closed subcategories of **Ord**($\mathcal{Q}$) instead of Cartesian closed ones. By the way, monoidal categories are also frequently investigated as mathematical structures both in category theory and theoretical computer science. In fact, some non-idempotent (non-classical) logics are closely related to monoidal closed categories, for instance, Barr's $*$-autonomous categories provide models for Girard's linear logic [1].

Let $\mathcal{Q} = (Q, \&)$ be a complete residuated lattice. In this paper, we take the notion of liminf completeness of Wagner [25,26] for the quantitative version of directed completeness. This notion specializes to the notion of Yoneda completeness in [3,16,24] for generalized metric spaces if $\mathcal{Q} = ([0,\infty]^{\rm op}, +)$. The category **Liminf**($\mathcal{Q}$) of liminf complete $\mathcal{Q}$-ordered sets and liminf continuous maps is a natural candidate for the role of **DCPO** in the fuzzy setting. We are concerned with the Cartesian and monoidal closedness of this category.

As shall be shown, the category **Liminf**($\mathcal{Q}$) is seldom Cartesian closed. In fact, if $\mathcal{Q} = ([0,1], \&)$ with & being a continuous t-norm, then **Liminf**($\mathcal{Q}$) is Cartesian closed if and only if & is the t-norm min. As for monoidal closedness, the result obtained here looks more promising: if the semigroup operation & in $\mathcal{Q} = (Q, \&)$ distributes over non-empty meets, then **Liminf**($\mathcal{Q}$) is monoidal closed. In particular, if $\mathcal{Q} = ([0,1], \&)$ with & being a left continuous t-norm, then **Liminf**($\mathcal{Q}$) is monoidal closed if and only if & is continuous.

These results exhibit a close relationship between properties of the truth-value table $\mathcal{Q}$ and categorical properties of fuzzy orders. Many things remain to be discovered in this respect.

The content is arranged as follows. Section 2 recalls some basic notions of complete residuated lattices and $\mathcal{Q}$-ordered sets; Section 3 focuses on the monoidal closedness of **Liminf**($\mathcal{Q}$); Section 4 is concerned with the Cartesian closedness of **Liminf**($\mathcal{Q}$).

The reader is referred to the classic text [22] for ideas and results in category theory.

2. Liminf complete orders valued in a complete residuated lattice

A complete residuated lattice $\mathcal{Q}$ is a pair $(Q, \&)$ such that

(i) Q is a complete lattice with a bottom element 0 and a top element 1;
(ii) $(Q, \&, 1)$ is commutative monoid; and
(iii) & distributes over arbitrary joins, i.e., $q \,\&\, \bigvee_{i\in I} p_i = \bigvee_{i\in I}(q \,\&\, p_i)$ for all q and p_i in Q.

The complete residuated lattices considered in this paper are often required to satisfy the following distributive law:

$$\text{(RC)} \quad p \,\&\, \bigwedge_{i\in J} q_i = \bigwedge_{i\in J}(p \,\&\, q_i)$$

for all $p, q_i \in Q$ whenever the index set J is non-empty. That is, & distributes over non-empty meets.

Example 2.1.

(1) Every frame $(H, \wedge)$ is a complete residuated lattice that satisfies the distributive law (RC).
(2) Complete BL-algebras [11], complete MV-algebras in particular, are all complete residuated lattices. Each complete MV-algebra satisfies (RC), but, we do not know whether (RC) is satisfied by all complete BL-algebras.
(3) Let & be a left continuous t-norm on the unit interval $[0,1]$. Then $([0,1], \&)$ is a complete residuated lattice. Further, $([0,1], \&)$ satisfies (RC) if and only if & is a continuous t-norm.
(4) $([0,\infty]^{\rm op}, +)$ is a complete residuated lattice that satisfies (RC). The correspondence $x \mapsto {\rm e}^{-x}$ is an isomorphism from $([0,\infty]^{\rm op}, +)$ to $([0,1], \cdot)$, where $\cdot$ denotes the product t-norm.

Let $\mathcal{Q} = (Q, \&)$ be a complete residuated lattice. A $\mathcal{Q}$-order (or a fuzzy partial order) [2] on a set A is a map $R : A \times A \longrightarrow Q$ such that

(1) (reflexivity) $1 \le R(x,x)$ for all $x \in A$;
(2) (transitivity) $R(y,z) \,\&\, R(x,y) \le R(x,z)$ for all $x, y, z \in A$;
(3) (antisymmetry) $1 \le R(x,y) \wedge R(y,x)$ implies that $x = y$ for all $x, y \in A$.