Available online at www.sciencedirect.com

ScienceDirect

Fuzzy Sets and Systems 282 (2016) 99–114

FUZZY sets and systems

www.elsevier.com/locate/fss

Fuzzy pseudo-norms and fuzzy F-spaces

Sorin Nădăban

Aurel Vlaicu University of Arad, Department of Mathematics and Computer Science, Str. Elena Drăgoi 2, RO-310330 Arad, Romania

Received 25 June 2013; received in revised form 10 December 2014; accepted 20 December 2014

Available online 24 December 2014

Abstract

In the present paper we firstly introduce the notion of fuzzy pseudo-norm, then we extend, improve and complete the results obtained by T. Bag and S.K. Samanta for fuzzy norms, in the fuzzy pseudo-norms context. Lastly, we introduce and discuss the notions of fuzzy F-norm and fuzzy F-space. By means of several auxiliary results, we obtain a characterization of metrizable topological linear spaces in terms of fuzzy F-norm.

Keywords: Fuzzy pseudo-norm; Fuzzy F-norm; Fuzzy metric space; Fuzzy metrizable topological linear space

1. Introduction

The models we work with, mathematical in their nature, must arrange themselves into pre-existing structures. In functional analysis, the fundamental structure is that of a topological linear space, but its degree of generality is much too high. Consequently, many of the important results in functional analysis have been obtained on Banach spaces (complete normed linear spaces). A significant number of familiar and useful topological linear spaces have a natural metric structure and are complete. Nevertheless, this metric does not come from a norm. These are Fréchet spaces, a term introduced by S. Banach in honour of M. Fréchet. Today, the term Fréchet space is used for a particular class of metrizable topological linear spaces, namely for the locally convex ones, while the term F- space is used for complete metrizable topological linear spaces. The topology of an F-space can be given by means of an F-norm.

The foundations of fuzzy functional analysis were laid by A.K. Katsaras, who studied fuzzy topological linear spaces in his works [10,11]. Moreover, A.K. Katsaras was the first to introduce the notion of fuzzy norm – of a Minkowski type – on a linear space, associated to an absolutely convex (convex and balanced) absorbing fuzzy set. From A.K. Katsaras onward, many mathematicians have proposed several notions of fuzzy norm from different points of view. Thus, in 1992, C. Felbin [5] introduced the idea of fuzzy norm on a linear space by assigning a fuzzy real number to each element of the linear space. In 2003, following S.C. Cheng and J.N. Mordeson [4], T. Bag and S.K. Samanta [2] proposed another concept of fuzzy norm. The Bag–Samanta fuzzy norm type has proved to be the most adequate of all, even though it can be still polished, simplified, improved or generalized (see [1,7,14]). We must

E-mail address: snadaban@gmail.com.

http://dx.doi.org/10.1016/j.fss.2014.12.010

note that T. Bag and S.K. Samanta, in paper [3], have continued the study of fuzzy norms, obtaining relations between fuzzy norms on one hand, and the associated family of α-norms, on the other hand. They introduced different types of fuzzy continuity and fuzzy boundedness of linear operators. In addition to this, T. Bag and S.K. Samanta have established some principles of functional analysis in fuzzy settings, which represent a foundation for the development of fuzzy functional analysis, based on their approach. However, the notion of Šerstnev space led to a concept of fuzzy norm recently introduced by I. Goleţ [7], in 2010. Goleţ's notion generalizes to continuous t-norm the concept of fuzzy norm defined by S.C. Cheng and J.N. Mordeson [4]. In 2010, C. Alegre and S. Romaguera [1] proposed the term of fuzzy quasi-norm on a real linear space and obtained characterizations of metrizable topological linear spaces.

Motivated by the works of T. Bag and S.K. Samanta on fuzzy normed linear spaces, the aim of this paper is to introduce the notion of fuzzy pseudo-norm and to extend and complete the results obtained by T. Bag and S.K. Samanta, for fuzzy norms, in the context of fuzzy pseudo-norms. Finally, the concept of fuzzy F-norm is introduced and used for characterization of metrizable topological linear spaces, by replacing fuzzy norms of type $(N, *_L)$ or $(N, \cdot)$ as stated in [1], with fuzzy F-norms $(F, \min)$. By introducing the concept of fuzzy F-space, we will be able to obtain, in a further paper, fuzzy versions for classical principles of functional analysis in this much more general context, extending T. Bag's and S.K. Samanta's results in paper [3], as well as the theorems obtained by I. Sadeqi and F.S. Kia [17] in 2009.

The structure of the paper is as follows: after the preliminary section, in Section 3, the concept of fuzzy pseudo-norm is introduced. Some of the conditions of Bag–Samanta's notion are modified (N6 and N7) or generalized (N3). Thus, any pseudo-norm induces, in a natural way, a fuzzy pseudo-norm (Example 3.3). Some decomposition theorems for fuzzy pseudo-norms into a family of pseudo-norms are obtained. The convergence in fuzzy pseudo-normed linear spaces is studied in Section 4. The concept of fuzzy F-norm is introduced in Section 5. We establish some decomposition theorems for fuzzy F-norms. In Theorem 5.8 we prove that a topological linear space is fuzzy metrizable if and only if it is metrizable, result obtained by V. Gregori and S. Romaguera [8] in 2000, without assuming a linear space structure. Also Theorem 5.8 gives a characterization of metrizable topological linear spaces.

2. Preliminaries

Definition 2.1. (See [13].) The pair (X, M) is said to be a fuzzy metric space if X is an arbitrary set and M is a fuzzy set in $X \times X \times [0, \infty)$ satisfying the following conditions:

(M1) $M(x, y, 0) = 0$, $(\forall)x, y \in X$;
(M2) $(\forall)x, y \in X$, $x = y$ if and only if $M(x, y, t) = 1$ for all $t > 0$;
(M3) $M(x, y, t) = M(y, x, t)$, $(\forall)x, y \in X$, $(\forall)t > 0$;
(M4) $M(x, z, t + s) \geq \min\{M(x, y, t), M(y, z, s)\}$, $(\forall)x, y, z \in X$, $(\forall)t, s > 0$;
(M5) $(\forall)x, y \in X$, $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is left continuous and $\lim_{t\to\infty} M(x, y, t) = 1$.

Remark 2.2. We note that the notion of a fuzzy metric, as given by I. Kramosil and J. Michálek, is more general that the one given in Definition 2.1. In fact, the condition (M4) is

$$M(x, z, t + s) \geq M(x, y, t) * M(y, z, s), \quad (\forall)x, y, z \in X, \ (\forall)t, s > 0,$$

where $*$ is a continuous t-norm (see [19]). The basic examples of continuous t-norms are minimum, usual multiplication denoted by $\cdot$ and the Lukasiewicz t-norm $*_L$ defined by $a *_L b = \max\{a + b - 1, 0\}$, but, in this paper we will work only with $* = \min$.

Our basic reference for fuzzy metric space and related structures is [9], while for t-norms, is [12].

Definition 2.3. (See [2].) Let X be a linear space over a field $\mathbb{K}$ (where $\mathbb{K}$ is the space of real numbers $\mathbb{R}$ or the space of complex numbers $\mathbb{C}$). A fuzzy set N in $X \times \mathbb{R}$ is called a fuzzy norm on X if it satisfies:

(N1) $N(x, t) = 0$, $(\forall)t \leq 0$;
(N2) $N(x, t) = 1$, $(\forall)t \in \mathbb{R}_+^*$ if and only if $x = 0$, where $\mathbb{R}_+^*$ denotes the set of all positive reals;