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Fuzzy Sets and Systems 282 (2016) 115–130

FUZZY sets and systems

www.elsevier.com/locate/fss

Approximation properties in fuzzy normed spaces

Keun Young Lee [1]

Institute for Ubiquitous Information Technology and Applications, Konkuk University, Seoul, 143-701, Republic of Korea

Received 5 March 2012; received in revised form 2 February 2015; accepted 3 February 2015

Available online 9 February 2015

Abstract

The paper is concerned with approximation properties which are defined in fuzzy normed spaces. First, we introduce the approximation property and the bounded approximation property. Next, we provide two main examples: First example is a fuzzy normed space that does not have the approximation property and second example is a fuzzy normed space with the approximation property that does not have the bounded approximation property. Finally, we provide characterizations of approximation properties in fuzzy normed spaces.

Keywords: Fuzzy norm; Fuzzy bounded linear operator; Approximation property; Bounded approximation property

1. Introduction

Approximation properties, which were systematically investigated by Grothendieck [16], are one of the most important properties in Banach space theory. Let X be a Banach space. X has the *approximation property* (AP) if for every compact K and $\varepsilon > 0$, there is a bounded finite rank operator $T : X \to X$ such that $\|T(x) - x\| < \varepsilon$ for all $x \in K$, i.e. I_X –the identity operator on X– can be approximated by finite rank operators uniformly on compact sets. Also recall that X has the *bounded approximation property* (BAP) if for every compact K and $\varepsilon > 0$, there is a bounded finite rank operator $T : X \to X$ with $\|T\| \le \lambda$ such that $\|T(x) - x\| < \varepsilon$ for all $x \in K$ for some $\lambda > 0$.

The theory of approximation property plays an important role in the research about the structure of Banach spaces of infinite dimensions along with a study of Schauder basis. Especially, a study on relation between approximation properties and Schauder basis has been conducted for a long time (see, [10,25,27]). For instance, we recall that a separable Banach space X has BAP if and only if X is a complemented subspace of a Banach space with a Schauder basis [23]. On the other hand, Schauder bases are applied to fractal functions and the differential problem by constructing an approximation of the load function which is inverse of a given function (see, [5,24]). Especially, the approximation suggested by [24] is similar to BAP's simple characterization in separable case, i.e. there exists a sequence (T_n) of finite rank operators on X such that $\lim_{n\to\infty} \|T_n(x) - x\| = 0$ for $x \in X$. Since BAP or AP is strictly weaker property

E-mail address: northstar@kaist.ac.kr.

[1] Tel.: +82 2 450 3476.

http://dx.doi.org/10.1016/j.fss.2015.02.003

than having Schauder basis, we hope that approximation properties bring more general results on applications along with the differential problem than those of having Schauder basis.

Since Katsaras [17] first introduced the notion of fuzzy norm on a vector space, many researches on fuzzy normed spaces have been conducted. Felbin [11] introduced an idea of fuzzy normed space and defined fuzzy norm of a linear operator from a fuzzy normed linear space to another fuzzy normed linear space. Also Cheng and Mordeson [6] gave another concept of a fuzzy norm on a linear space whose associated metric is of Kramosil [19] and Michalek type [9]. Cheng and Mordeson [6], Bag and Samanta [1] studied some properties of finite dimensional fuzzy normed spaces. Further, recently, Bag and Samanta [2] introduced another type of boundedness of linear operators and fuzzy dual spaces and proved analogues of fundamental theorems in functional analysis. Then, Lael and Nourouzi [20] introduced fuzzy compact linear operators and established some results related to this notion.

Recently, the concept of the approximation property in fuzzy normed spaces was introduced by Yilmaz [28]. He defined the weak fuzzy approximation property and strong fuzzy approximation property by using a fuzzy norm and provided some examples and basic results about these. However, Yilmaz's definitions, since approximating operators do not need to be continuous in sense of fuzzy operators, have some limitations to investigate structures of fuzzy normed spaces by using these. Hence, to obtain various results for approximation property in fuzzy normed spaces, we need to modify Yilmaz's definitions.

Our research is motivated by the most famous questions of the approximation property and the bounded approximation property for Banach spaces following as.

Questions.

(1) *Does every Banach space have AP?*
(2) *Does AP imply BAP?*
(3) *What are the various variants of AP and BAP?*

Question (1) that appeared in Banach's book [4] and remained open for forty years was solved (in the negative) by Enflo [10]. Question (2) was solved by (also, in the negative) Figiel and Johnson [12]. Solution to Question (3) was initiated by Grothendieck [16].

Our main interest in this paper is to find solutions to analogous questions of approximation properties in fuzzy normed spaces on our modified setting. That is, we put a bridge between classical Banach spaces and fuzzy normed spaces from approximation properties point of view. Our idea of approximation properties in fuzzy normed spaces contributes to theory of fuzzy normed spaces as follows. First, we provide modified definitions of approximation properties in fuzzy normed spaces enough to obtain various results for these properties. Second, we give all answers to Questions (1), (2) and (3) related approximation properties in fuzzy normed spaces. Finally, we develop various topological methods in the space of strongly fuzzy continuous operators to provide the variants of approximation properties in fuzzy normed spaces.

Our paper is organized as follows. In Section 2, we fix notation and introduce basic facts of fuzzy normed spaces and fuzzy linear operators. In Section 3, we define approximation properties and bounded approximation properties in fuzzy normed spaces and justify our definitions. Furthermore, we provide main problems that we deal with in this paper. In Section 4, we provide sufficient and necessary conditions to have approximation property in fuzzy normed spaces and two main examples which one is a fuzzy normed space that does not have the approximation property and the other is a fuzzy normed space with the approximation property that does not have the bounded fuzzy approximation property. In Section 5, we investigate the space of strongly fuzzy bounded linear operators and its topologies. In the final section, by using results in Section 5, we establish characterizations of the bounded approximation property in fuzzy normed spaces as solutions to our problem.

2. Notations and preliminaries

Definition 2.1. (See [26].) A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-norm if $*$ satisfies the following conditions: