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On the uniform boundedness theorem in fuzzy quasi-normed spaces

Carmen Alegre [*,1], Salvador Romaguera [1]

Instituto Universitario de Matemática Pura y Aplicada, Universitat Politècnica de València, Camí de Vera s/n, 46022 Valencia, Spain

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Abstract

We prove that a family of continuous linear operators from a fuzzy quasi-normed space of the half second category to a fuzzy quasi-normed space is uniformly fuzzy bounded if and only if it is pointwise fuzzy bounded. This result generalizes and unifies several well-known results; in fact, the classical uniform boundedness principle, or Banach–Steinhauss theorem, is deduced as a particular case. Furthermore, we establish the relationship between uniform fuzzy boundedness and equicontinuity which allows us to give a uniform boundedness theorem in the class of paratopological vector spaces. The classical result for topological vector spaces is deduced as a corollary.

Keywords: Fuzzy quasi-norm; Fuzzy norm; Uniformly fuzzy bounded; Equicontinuous; Quasi-normed space; Paratopological vector space

1. Introduction

It is well known that the uniform boundedness principle, or Banach–Steinhauss theorem, is a crucial result in functional analysis. It essentially establishes that a family of continuous linear operators from a Banach space to a normed space is uniformly bounded if and only if it is pointwise bounded (in operator norm). Although its original proofs [10,20] were constructive, in many textbooks we can find a proof based on the use of the Baire category theorem (see e.g. [6, Theorem 6.14]). In fact, this method of proof actually shows the following more general result (recall that there exist non-Banach normed spaces that are of the second category [21]).

Theorem 1. *Let $(X, \|.\|_1)$ and $(Y, \|.\|_2)$ be two normed spaces such that $(X, \|.\|_1)$ is of the second category. If $\mathcal{F}$ is a family of continuous linear operators from $(X, \|.\|_1)$ to $(Y, \|.\|_2)$ such that for each $x \in X$ there exists $b_x > 0$ with $\|f(x)\|_2 \le b_x$ for all $f \in \mathcal{F}$, then there exists $b > 0$ such that*

$$\sup\left\{\|f(x)\|_2 : \|x\|_1 \le 1\right\} \le b,$$

for all $f \in \mathcal{F}$.

[*] Corresponding author.
E-mail addresses: calegre@mat.upv.es (C. Alegre), sromague@mat.upv.es (S. Romaguera).
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Next we state a generalization of Theorem 1 for topological vector spaces which we can find for instance in [24, Theorem 13.28] (recall that a family of continuous linear operators from a normed space of the second category to a normed space is equicontinuous if and only if it is uniformly bounded).

Theorem 2. *Let (X, τ) and (Y, τ') be two topological vector spaces such that (X, τ) is of the second category. Then, a family of continuous linear operators from (X, τ) to (Y, τ') is equicontinuous if and only if it is pointwise bounded.*

Recently, Alegre, Romaguera and Veeramani [5, Theorem 2.6] proved the following asymmetric generalization of Theorem 1.

Theorem 3. *Let (X, p) and (Y, q) be two asymmetric normed spaces such that (X, p) is of the half second category. If $\mathcal{F}$ is a family of continuous linear operators from (X, p) to (Y, q) such that for each $x \in X$ there exists $b_x > 0$ with $q(f(x)) \le b_x$ for all $f \in \mathcal{F}$, then there exists $b > 0$ such that*

$$\sup\{q(f(x)) : p(x) \le 1\} \le b,$$

for all $f \in \mathcal{F}$.

On the other hand, the study of fuzzy (quasi-)normed spaces has received a certain attention in the last years (see e.g. [4,7–9,18,26–28, etc.]). An objective of these investigations is the study of fundamental concepts and properties of functional analysis from a fuzzy set point of view.

In this context, and motivated by the fact that the class of (quasi-)normed spaces is strictly contained in the class of fuzzy (quasi-)normed spaces [4], the following interesting question arises in a natural way: Is it possible to give a type of uniform boundedness theorem in the framework of fuzzy (quasi-)normed spaces which simultaneously generalizes Theorems 1 and 3 above?

Here we shall give a positive answer to this question. Related to it we recall that Bag and Samanta stated in [7] a uniform boundedness type theorem for a special class of fuzzy normed spaces using in their proof the classical theorem for normed spaces. Our results are given for a very general class of fuzzy quasi-normed spaces and allow us to generalize and unify the theorems for (asymmetric) normed spaces cited above. In particular, since every (quasi-)metrizable (para)topological vector space admits a structure of fuzzy (quasi-)normed space [4], we deduce a uniform boundedness theorem for paratopological vector spaces that was not known in the literature up to now and that extends Theorem 2 to the asymmetric framework. We do it by showing that in our fuzzy context the classical notion of equicontinuity is equivalent to uniform fuzzy boundedness.

2. Terminology and basic definitions

In this section we collect several well-known concepts, facts and examples from the theories of quasi-metric spaces, quasi-normed spaces, fuzzy quasi-metric spaces and fuzzy quasi-normed spaces which will be useful in the rest of the paper. Our basic references will be [11,15,19,23].

In the sequel, the letters $\mathbb{R}$ and $\mathbb{N}$ will denote the set of real numbers and the set of positive integer numbers, respectively.

Following the modern terminology, by a quasi-metric on a set X we mean a function $d : X \times X \to [0, \infty)$ such that for all $x, y, z \in X$: (i) $d(x, y) = d(y, x) = 0 \Leftrightarrow x = y$, and (ii) $d(x, y) \le d(x, z) + d(z, y)$.

Then, the pair (X, d) is called a quasi-metric space.

Each quasi-metric d on X induces a T_0 topology τ_d on X which has as a base the family of balls $\{B_d(x, r) : x \in X, r > 0\}$, where $B_d(x, r) = \{y \in X : d(x, y) < r\}$.

A topological space (X, τ) is said to be quasi-metrizable if there is a quasi-metric d on X such that $\tau = \tau_d$. In this case, we say that τ is a quasi-metrizable topology.

A quasi-norm on a real vector space X is a function $q : X \to [0, \infty)$ such that for all $x, y \in X$ and $r \ge 0$: (i) $q(x) = q(-x) = 0 \Leftrightarrow x = 0$, (ii) $q(rx) = rq(x)$, and (iii) $q(x + y) \le q(x) + q(y)$.

Then, the pair (X, q) is said to be a quasi-normed linear space or, simply, a quasi-normed space.

In [15] (see also [5,12,16]), quasi-norms are called asymmetric norms and quasi-normed spaces are called asymmetric normed spaces.