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Ensuring reliability of the weighting vector: Weak consistent pairwise comparison matrices

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Abstract

In the context of Pairwise Comparison Matrices (PCMs) defined over abelian linearly ordered group, $\odot$-consistency and $\odot$-transitivity represent a full coherence of the Decision Maker (DM) and the minimal logical requirement that DM's preferences should satisfy, respectively. Moreover, the $\odot$-mean vector $\boldsymbol{w}_{m_\odot}$ is proposed as weighting vector for the decision elements related to the PCM. In this paper, we investigate the effects of $\odot$-inconsistency of a $\odot$-transitive PCM on $\boldsymbol{w}_{m_\odot}$ and, in order to ensure its reliability as weighting vector, we provide the notion of weak $\odot$-consistency; it is weaker than $\odot$-consistency and stronger than $\odot$-transitivity, and ensures that vectors associated with a PCM, by means of a strictly increasing synthesis functional, are reliable for assigning a preference order on the set of related decision elements. The $\odot$-mean vector $\boldsymbol{w}_{m_\odot}$ is associated with a PCM by means of one of these functionals. Finally, we introduce an order relation on the rows of the PCM, that is a simple order if and only if the condition of weak $\odot$-consistency is satisfied; the simple order allows us to easily determine the actual ranking on the set of related decision elements.

Keywords: Multiple criteria evaluation; Abelian linearly ordered group; Pairwise comparison matrix; Weak $\odot$-consistency; Coherent priority vector

1. Introduction

The quantitative pairwise comparisons are a useful tool for estimating the relative weights on a set $X = \{x_1, x_2, ..., x_n\}$ of decision elements such as criteria or alternatives. Pairwise comparisons can be modeled by a quantitative preference relation on X:

$$\mathcal{A} : (x_i, x_j) \in X \times X \to a_{ij} = \mathcal{A}(x_i, x_j) \in \mathbb{R}$$

where a_{ij} quantifies the preference intensity of x_i over x_j. When the cardinality of X is small, $\mathcal{A}$ can be represented by the *Pairwise Comparison Matrix* (*PCM*)

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$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}. \tag{1}$$

In the literature, several kinds of PCMs are considered: if $a_{ij} \in]0, +\infty[$ represents a preference ratio, then $\mathcal{A}$ is a *multiplicative* preference relation and A a *multiplicative* PCM; if $a_{ij} \in \mathbb{R} =]-\infty, +\infty[$ represents a preference difference, then $\mathcal{A}$ is an *additive* preference relation and A an *additive* PCM; if $a_{ij} \in [0, 1]$ reflects a preference degree, then $\mathcal{A}$ is called fuzzy preference relation [14,19] and, as a consequence, A can be called *fuzzy* PCM.

If the DM is fully coherent when expresses his preferences, then the PCM satisfies the *consistency* condition; several kinds of consistency are proposed in literature: multiplicative consistency [23], additive consistency [1], fuzzy additive consistency (called additive transitivity in [25,15]), and fuzzy multiplicative consistency (called multiplicative transitivity in [15,24]).

Many authors have studied the problem of the inconsistency of a PCM, for instance: Saaty [22] proposes a consistency index defined in terms of the principal eigenvalue; Barzilai [1] proposes the relative error; Peláez and Lamata [20] propose a measure of consistency based on the determinant of the PCM; Bortot and Marques Pereira [5] propose a measure of dominance inconsistency in the framework of Choquet integration; Brunelli and Fedrizzi [6] present five axioms aimed at characterizing inconsistency indices.

In order to unify different approaches to PCMs and remove some drawbacks, in [8] the authors introduce PCMs defined over an abelian linearly ordered group (*alo-group*) $\mathcal{G} = (G, \odot, \leq)$, with identity element e (a comparison between an alo-group and a semiring, for dealing with PCMs, is performed in [7]).

In this general context, conditions of $\odot$-reciprocity and $\odot$-consistency are proposed. If $\mathcal{G}$ is divisible, then, by means of a mean operator $m_\odot$, $\odot$-consistency index $I_\mathcal{G}$ (see [13] for its properties) and $\odot$-mean vector $\boldsymbol{w}_{m_\odot}$ [10] are defined for measuring $\odot$-consistency and providing the weights for the alternatives/criteria, respectively. Under assumption of $\odot$-reciprocity, we set [11]:

$$x_i \succ x_j \Leftrightarrow a_{ij} > e, \qquad x_i \sim x_j \Leftrightarrow a_{ij} = e, \tag{2}$$

where $x_i \succ x_j$ and $x_i \sim x_j$ stand for "x_i is strictly preferred to x_j" and "x_i is indifferent to x_j", respectively, and

$$x_i \succsim x_j \Leftrightarrow (x_i \succ x_j \text{ or } x_i \sim x_j) \Leftrightarrow a_{ij} \geq e, \tag{3}$$

that stands for "x_i is weakly preferred to x_j".

Finally, in [11], $\odot$-transitivity, equivalent to transitivity of $\succsim$, ensures a rearrangement $(i_1, i_2, \cdots, i_n)$ of $\{1, 2, \cdots, n\}$ such that:

$$x_{i_1} \succsim x_{i_2} \succsim \ldots \succsim x_{i_n}; \tag{4}$$

equation (4) is called the *actual ranking* on X.

Under $\odot$-consistency condition, the weighting vector $\boldsymbol{w}_{m_\odot}$ provides a preference order on X equal to the actual ranking. Unfortunately, $\odot$-consistency is hard to reach in real situations; thus, it may happen that the weighting vector $\boldsymbol{w}_{m_\odot}$ provides a preference order on X different from the actual ranking.

This paper aims at dealing with the following research question:

RQ. *Let $A = (a_{ij})$ be a $\odot$-transitive PCM. Under which conditions weaker and easier to reach in real situations than $\odot$-consistency, does the weighting vector $\boldsymbol{w}_{m_\odot}$ provide a preference order on X equal to the actual ranking?*

We provide an answer to **RQ**, by introducing the *weak $\odot$-consistency* condition.

The remainder of this paper is organized as follows. Section 2 introduces preliminaries and notation useful in the sequel. Section 3 considers $\odot$-reciprocal PCMs, by focusing on $\odot$-transitive PCMs, $\odot$-consistent PCMs, and effects of $\odot$-inconsistency on the weighting vector $\boldsymbol{w}_{m_\odot}$. Section 4 introduces the weak $\odot$-consistency: Section 4.1 provides its characterizations; Section 4.2 shows that reaching and checking the weak $\odot$-consistency is easier than reaching and checking the $\odot$-consistency; Section 4.3 shows that the vectors associated with a weakly $\odot$-consistent PCM, by means of strictly increasing synthesis functionals, e.g. $\boldsymbol{w}_{m_\odot}$, provide a preference order on X equal to actual ranking. Section 5 provides concluding remarks and directions for future work.