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Necessary and sufficient conditions for fuzzy optimality problems [☆]

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Abstract

In this paper we define a new minimum concept for fuzzy optimization problems more general than those that exist in the literature. We find necessary optimality conditions based on a new fuzzy stationary point definition. And we prove that these conditions are also sufficient under new fuzzy generalized convexity notions.

Keywords: Fuzzy optimization; Differentiable fuzzy mappings; Fuzzy generalized convexity

1. Introduction

Since Zadeh [28] introduced the fuzzy number concept and Chang and Zadeh [7] proposed the fuzzy mapping notion, these notions have been widely studied by many authors. One of these research lines has been the fuzzy optimization. There are a lot of works on optimality conditions for fuzzy problems, which are a proof of the great interest that this topic arouses among researchers.

In classical optimization methods, it is well-known that the stationary point concept (the one that cancels the derivative) plays a crucial role as a necessary optimality condition for problems defined by differentiable functions, since it allows to identify the potential candidates to be optimums. So far, the fuzzy stationary points definitions [14, 24–26] are very restrictive from two points of view: one due to the derivative definition used, and the other due to the own definition.

The optimality conditions are based necessarily on a stationary point notion, which in turn is based on a derivative definition. Here you have the first difference of the work that we present. Other works demand H-differentiability or level-wise differentiability notions. There are references, [1,2,6], that demonstrate that these differentiability notions

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are very restrictive (as you can see in Example 1). So we use a differentiability notion more general and so more efficient that the previous ones used.

On the other hand, the stationary point notion that we define is clearly different from studied till now in the literature. In previous works, either necessary optimality conditions are not proved, or they are proved under restrictive conditions (comparable functions), or they are complicated conditions to check, [14,24–26]. So, it is important to introduce a new fuzzy stationary point definition more adequate. The optimality conditions that we prove, present conceptual and specially computational checking advantages, since we demonstrate that they are equivalent to simply check that zero belongs to an interval.

Another main part in optimization theory is establishing sufficient optimality conditions. It is also known that not all stationary points are optimal, so it is necessary to use some problem intrinsic properties for eliminating the non-optimal candidates. Convexity or generalized convexity hypotheses made on the functions that define the problem are some of these properties.

It is well-known that in classical optimization, convexity plays a central role in order to get sufficient conditions or to characterize the solutions set. Nevertheless in fuzzy environment it has been demonstrated, [17], that some differentiable convex classic function properties do not remain valid. Therefore it is necessary to define adequately new generalized convexity concepts for differentiable fuzzy functions, which allow us to characterize the fuzzy optima through the stationary points.

In fuzzy optimization different efforts have been made to define analogous concepts to the convexity in order to establish necessary and sufficient optimality conditions. The convex fuzzy mappings characterizations, with and without differentiability, other generalized convexity concepts and their applications to fuzzy nonlinear programming were studied by several authors (see, [9,14,19–23,25–27]). Examples can be found that demonstrate that these hypotheses are restrictive too (Example 7 presents a function that is not convex, nor invex, but it is strict pseudoinvex) and therefore the results that we prove in this paper are applicable to a wider range of functions.

The aim of this work is defining a more general stationary point notion, using a more general differentiability definition, that is applicable to a wider range of functions and that it is easier to check from the computational point of view too. We show that under those conditions, every optimum for a fuzzy mapping is a fuzzy stationary point for that mapping. We define new generalized convexity notions that allow us to prove that every fuzzy stationary point is an optimum. We will prove that the stationary points and generalized convexity concepts here defined generalize the classic ones, since they coincide when the functions are not fuzzy ones.

2. Preliminaries

We denote by $\mathcal{K}_C$ the family of all bounded closed intervals in $\mathbb{R}$, i.e.,

$$\mathcal{K}_C = \left\{ \left[\underline{a}, \bar{a}\right] \,/\, \underline{a}, \bar{a} \in \mathbb{R} \text{ and } \underline{a} \le \bar{a} \right\},$$

Given two intervals $A = [\underline{a}, \bar{a}]$ and $B = [\underline{b}, \bar{b}]$ we define the distance between A and B by

$$H(A, B) = \max \left\{ \left|\underline{a} - \underline{b}\right|, \left|\bar{a} - \bar{b}\right| \right\}.$$

It is well-known that $(\mathcal{K}_C, H)$ is a complete metric space [8].

A fuzzy set on $\mathbb{R}^n$ is a mapping $u : \mathbb{R}^n \to [0, 1]$. For each fuzzy set u, we denote $[u]^\alpha = \{x \in \mathbb{R}^n | u(x) \ge \alpha\}$ for any $\alpha \in (0, 1]$ its α-level sets. By *suppu* we denote the support of u, i.e. $\{x \in \mathbb{R}^n | \, u(x) > 0\}$. By $[u]^0$ we define the closure of *supp u*.

Definition 1. A compact and convex fuzzy set u on $\mathbb{R}^n$ is a fuzzy set with the following properties:

(1) u is normal, i.e. there exists $x_0 \in \mathbb{R}^n$ such that $u(x_0) = 1$;
(2) u is an upper semi-continuous function;
(3) $u(\lambda x + (1 - \lambda)y) \ge min\{u(x), u(y)\}, \; x, y \in \mathbb{R}^n, \; \lambda \in [0, 1]$;
(4) $[u]^0$ is compact.

Let $\mathcal{F}_C$ be the set of the all compact and convex fuzzy sets on $\mathbb{R}$. Obviously, if $u \in \mathcal{F}_C$, $[u]^\alpha$ is a nonempty compact and convex subset of $\mathbb{R}$ (denoted by $\left[\underline{u}_\alpha, \bar{u}_\alpha\right]$) for any $\alpha \in [0, 1]$. So, if $u \in \mathcal{F}_C$ we say that u is a fuzzy interval. If a