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The ⊛-composition of fuzzy implications: Closures with respect to properties, powers and families

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Abstract

Recently, Vemuri and Jayaram proposed a novel method of generating fuzzy implications from a given pair of fuzzy implications. Viewing this as a binary operation ⊛ on the set $\mathbb{I}$ of fuzzy implications they obtained, for the first time, a monoid structure $(\mathbb{I}, \circledast)$ on the set $\mathbb{I}$. Some algebraic aspects of $(\mathbb{I}, \circledast)$ had already been explored and hitherto unknown representation results for the Yager's families of fuzzy implications were obtained in [53] (N.R. Vemuri and B. Jayaram, Representations through a monoid on the set of fuzzy implications, fuzzy sets and systems, 247 (2014) 51–67). However, the properties of fuzzy implications generated or obtained using the ⊛-composition have not been explored. In this work, the preservation of the basic properties like *neutrality, ordering and exchange principles*, the functional equations that the obtained fuzzy implications satisfy, the powers w.r.t. ⊛ and their convergence, and the closures of some families of fuzzy implications w.r.t. the operation ⊛, specifically the families of (S, N)-, R-, f- and g-implications, are studied. This study shows that the ⊛-composition carries over many of the desirable properties of the original fuzzy implications to the generated fuzzy implications and further, due to the associativity of the ⊛-composition one can obtain, often, infinitely many new fuzzy implications from a single fuzzy implication through self-composition w.r.t. the ⊛-composition.

Keywords: Fuzzy implication; Basic properties; Functional equations; Self-composition; Closure; (SN)-implications; R-implications; f-implications; g-implications

1. Introduction

Fuzzy implications, along with triangular norms (t-norms, in short) form the two most important fuzzy logic connectives. They are a generalisation of the classical implication and conjunction, respectively, to multivalued logic and play an equally important role in fuzzy logic as their counterparts in classical logic.

Fuzzy implications on the unit interval [0, 1] are defined as follows.

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Table 1
Examples of fuzzy implications (cf. Table 1.3 in [4]).

Name	Formula
Łukasiewicz	$I_{\mathbf{LK}}(x,y) = \min(1, 1-x+y)$
Gödel	$I_{\mathbf{GD}}(x,y) = \begin{cases} 1, & \text{if } x \le y \\ y, & \text{if } x > y \end{cases}$
Reichenbach	$I_{\mathbf{RC}}(x,y) = 1 - x + xy$
Kleene-Dienes	$I_{\mathbf{KD}}(x,y) = \max(1-x, y)$
Goguen	$I_{\mathbf{GG}}(x,y) = \begin{cases} 1, & \text{if } x \le y \\ \frac{y}{x}, & \text{if } x > y \end{cases}$
Rescher	$I_{\mathbf{RS}}(x,y) = \begin{cases} 1, & \text{if } x \le y \\ 0, & \text{if } x > y \end{cases}$
Yager	$I_{\mathbf{YG}}(x,y) = \begin{cases} 1, & \text{if } x = 0 \text{ and } y = 0 \\ y^x, & \text{if } x > 0 \text{ or } y > 0 \end{cases}$
Weber	$I_{\mathbf{WB}}(x,y) = \begin{cases} 1, & \text{if } x < 1 \\ y, & \text{if } x = 1 \end{cases}$
Fodor	$I_{\mathbf{FD}}(x,y) = \begin{cases} 1, & \text{if } x \le y \\ \max(1-x, y), & \text{if } x > y \end{cases}$
Least FI	$I_{\mathbf{0}}(x,y) = \begin{cases} 1, & \text{if } x = 0 \text{ or } y = 1 \\ 0, & \text{if } x > 0 \text{ and } y < 1 \end{cases}$
Greatest FI	$I_{\mathbf{1}}(x,y) = \begin{cases} 1, & \text{if } x < 1 \text{ or } y > 0 \\ 0, & \text{if } x = 1 \text{ and } y = 0 \end{cases}$
Most Strict	$I_{\mathbf{D}}(x,y) = \begin{cases} 1, & \text{if } x = 0 \\ y, & \text{if } x > 0 \end{cases}$

Definition 1.1. (See [4], Definition 1.1.1 & [27,16].) A function $I: [0,1]^2 \to [0,1]$ is called a *fuzzy implication* if it satisfies, for all $x, x_1, x_2, y, y_1, y_2 \in [0,1]$, the following conditions:

$$\text{if } x_1 \le x_2, \quad \text{then } I(x_1, y) \ge I(x_2, y), \text{ i.e., } I(\cdot, y) \text{ is decreasing,} \tag{I1}$$

$$\text{if } y_1 \le y_2, \quad \text{then } I(x, y_1) \le I(x, y_2), \text{ i.e., } I(x, \cdot) \text{ is increasing,} \tag{I2}$$

$$I(0,0) = 1, \qquad I(1,1) = 1, \qquad I(1,0) = 0. \tag{I3}$$

The set of fuzzy implications will be denoted by $\mathbb{I}$.

From Definition 1.1, it is clear that a fuzzy implication, when restricted to $\{0,1\}$, coincides with the classical implication. Table 1 (see also, Table 1.3 in [4]) lists some examples of basic fuzzy implications.

Fuzzy implications play an important role in approximate reasoning, fuzzy control, decision theory, control theory, expert systems, fuzzy mathematical morphology, image processing, etc. – see for example [9,11,21,22,49,54,56,57] or the recent monograph exclusively devoted to fuzzy implications [4].

The different generation methods of fuzzy implications can be broadly classified into the following three categories, viz,

(i) From binary functions on $[0,1]$, typically other fuzzy logic connectives, viz., (S,N)-, R-, QL-implications (see [4]),
(ii) From unary functions on $[0,1]$, typically monotonic functions, for instance, Yager's f-, g-implications (see [56]), or from fuzzy negations [7,20,35,46],
(iii) From fuzzy implications (see [3,6,12,13,15,19,39,46]).

1.1. Motivation for this work

Obtaining fuzzy implications from given fuzzy implications, the third method listed above, can be further subdivided into approaches that are either generative or constructive. By *generative* methods, we refer to those works which propose a closed form formula for obtaining new fuzzy implications from given ones, often with the help of