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ScienceDirect
Fuzzy Sets and Systems 275 (2015) 110–129
FUZZY sets and systems
www.elsevier.com/locate/fss

Rewriting systems over similarity and generalized pseudometric spaces and their properties

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Received 9 February 2013; received in revised form 1 November 2014; accepted 3 November 2014

Available online 6 November 2014

Abstract

We present a study of confluence and related properties of fuzzy relations defined over similarity spaces. The ordinary confluence is an essential property of relations connected to the idea of rewriting and substituting which appears in abstract rewriting systems. This paper is a continuation of our previous paper, where we have introduced analogous notions related to substitutability in graded setting using residuated lattices as structures of truth degrees, leaving the ordinary notions a particular case when the underlying structure is the two-valued Boolean algebra. In this paper, we further extend our previous results by developing the notions of confluence and related properties respecting a given similarity relation. We also present definitions of confluence and related properties of relations on a generalized pseudometric spaces. Using the well-known link between generalized pseudometrics and similarities, we describe the connection of the notions defined on generalized pseudometric spaces to the corresponding notions on similarity spaces. The introduced notions are further illustrated by example of rewriting based on graded if–then rules.

Keywords: Confluence; Estimation; Fuzzy relations; Similarity spaces; Generalized pseudometric spaces; Residuated lattices

1. Introduction and problem setting

The confluence is a crucial property of binary relations which are used to model substitutions in the theory of abstract rewriting systems. The classical notions of confluence, as well as termination, and other related properties of binary relations have already been the subject of extensive research. Besides their theoretical importance, the notions were also applied in various fields of computer science. For instance, term rewriting systems can be used as a theoretical background for logic and functional programming [16], various logical deductive systems can be formalized by rewriting systems, and term rewriting plays an important role in algebraic specification of abstract types. In addition to its basic form, term rewriting has been extended in various ways to incorporate rule priorities [3], conditional rewriting [13,21], nominal rewriting [17], etc. A useful overview of abstract rewriting systems can be

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http://dx.doi.org/10.1016/j.fss.2014.11.001
0165-0114/

found in [2] and [31]. The topics studied in this paper are also related to sensitivity analysis in fuzzy relational systems [4,6,26]. In this section, we present a survey of utilized notions and a motivation for our investigation.

Reduction and confluence The notions of reduction and confluence can be introduced as follows. Let R be a binary relation on a set X and assume that $\langle \mathfrak{x}, \mathfrak{y} \rangle \in R$ means that one may substitute $\mathfrak{y}$ for $\mathfrak{x}$. The substitution "$\mathfrak{y}$ for $\mathfrak{x}$" can be informally explained so that, from a certain point of view, whenever $\mathfrak{x}$ does a certain job, $\mathfrak{y}$ does it as well. An element $\mathfrak{x} \in X$ is called reducible if $\langle \mathfrak{x}, \mathfrak{y} \rangle \in R$ for some $\mathfrak{y} \in Y$; otherwise, $\mathfrak{x}$ is called irreducible. By a reduction we mean any sequence $\mathfrak{x}_0, \ldots, \mathfrak{x}_n$ such that $\langle \mathfrak{x}_{i-1}, \mathfrak{x}_i \rangle \in R$ $(i = 1, \ldots, n)$. In this case, $\mathfrak{x}_0$ is said to be reducible to $\mathfrak{x}_n$. Relation R is called confluent whenever $\mathfrak{x}$ is reducible to both $\mathfrak{y}$ and $\mathfrak{y}'$ then there is some $\mathfrak{z}$ such that both $\mathfrak{y}$ and $\mathfrak{y}'$ are reducible to $\mathfrak{z}$.

Graded substitutability The basic motivation of our investigation is the fact that there are natural examples where the notion of substitutability is inherently fuzzy rather than crisp. Therefore, we look at substitutability and the related properties from the point of view of fuzzy logic [4,18,19] and fuzzy set theory [32]. For instance, given a collection of soft constraints (i.e., constraints that can be matched to a degree including intermediate degrees so that not only yes/no matches are taken into account) and a structure which is a model (i.e., satisfies all the constraints), we can think of replacing the model by another structure which does not fully satisfy all the constraints but is significantly cheaper than the original model. In such a case, the process of replacing can be seen as an approximate rewriting process: an element is replaced by another one which does the job sufficiently well. Naturally, there may be several choices for the replacement with different properties and thus it makes sense to define a graded relation expressing a degree to which "(model) M_1 can be replaced by (model) M_2" (a particular example of this type is given later in Section 5). It is therefore appealing to consider and explore properties related to graded substitutability which may be important for the rewriting process, e.g., graded confluence. The initial study of graded substitutability and related notions is presented in [7].

Substitutability and similarity spaces In the present paper, we elaborate on the results from [7] and extend the utilized notions by equipping the universe of discourse by a similarity relation (i.e., a fuzzy relation which is reflexive, symmetric, and transitive) prescribing degrees to which elements are similar. Therefore, the definition of a degree to which "$\mathfrak{y}$ is substitutable for $\mathfrak{x}$" that has been introduced in the earlier paper [7] can be extended so that the similarity is taken into account. Namely, we can define a degree to which "there are $\mathfrak{z}_1$ and $\mathfrak{z}_2$ such that $\mathfrak{x}$ is similar to $\mathfrak{z}_1$ and $\mathfrak{z}_2$ is substitutable for $\mathfrak{z}_1$ and $\mathfrak{z}_2$ is similar to $\mathfrak{y}$" as a more general degree of substitutability with respect to the underlying similarity. Clearly, such an approach can yield more natural results then the original approach especially in cases where there is no $\mathfrak{y}$ substitutable for $\mathfrak{x}$ but there are some $\mathfrak{z}_1$ and $\mathfrak{z}_2$ which are (very) similar to $\mathfrak{x}$ and $\mathfrak{y}$ respectively such that $\mathfrak{z}_2$ is substitutable for $\mathfrak{z}_1$ to a high degree.

Therefore, we will introduce notions related to substitutability given by a fuzzy relations defined on a similarity space (a set equipped with a similarity fuzzy relation, see [4]) instead of just a set of elements. From the theoretical point of view, dealing with substitutability issues on similarity spaces should be interesting from several points of view. In addition to the above-mentioned motivation, the extension should also be relevant in the context of general algebra and term rewriting. For instance, in [8,11,12] it is shown that the fundamental notions and results on the equational reasoning can be accommodated to the requirement of respecting in a natural way an underlying similarity relation on the universe set. Given an underlying similarity, one may require (or find out) two terms to be substitutable only to some degree which may result from the fact that they always evaluate to elements which are similar but not identical. Hence, the investigation of graded substitutability on similarity spaces can bring new insight into the graded equational reasoning.

Interestingly, the idea of rewriting modulo an equivalence relation appeared already in [30] and is further investigated in the seminal paper [20] by Huet. Our generalization is technically different from several viewpoints. First, the former approaches deal with ordinary rewriting and ordinary equivalences whereas our approach is developed over general complete residuated lattices as structures of truth degrees. Second, even in the crisp case when our similarity relations become ordinary equivalences, our approach is different in that we allow to substitute equivalent elements after each step of rewriting whereas [30] does not. Indeed, [30] and the later papers including [28] (cf. also [1,29]) are interested in whether two equivalent elements are rewritten to two equivalent results which is a different issue than that addressed in our paper.