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A representation of fuzzy numbers ☆

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Abstract

Each fuzzy number can be written in a unique way as the sum of a symmetric and a skew fuzzy number. These two kinds of fuzzy numbers correspond bijectively to certain functions on the unit interval that are of bounded variation. These bijections give rise to a representation of fuzzy numbers by pairs of functions of bounded variation. As application, mathematical structures on the set of functions of bounded variation are transported to that of fuzzy numbers, making the set of fuzzy numbers into a complete metric space, a commutative semiring, and a commutative residuated lattice.

Keywords: Fuzzy number; Symmetric fuzzy number; Skew fuzzy number; Function of bounded variation; Semiring; Complete metric space

1. Introduction and preliminaries

In 1965, Zadeh [32] introduced the concept of fuzzy sets. Fuzzy numbers are a kind of fuzzy sets that are of particular importance in fuzzy set theory. A fuzzy number [12] is a function $u : \mathbb{R} \longrightarrow [0, 1]$ satisfying the following conditions:

(1) u is normal, that is, there is a real number t_0 such that $u(t_0) = 1$;

(2) u is compactly supported, that is, the closure of $\{t \in \mathbb{R} : u(t) > 0\}$ is bounded;

(3) u is convex, that is, $r \le t \le s$ implies $\min\{u(r), u(s)\} \le u(t)$ for all $r, s, t \in \mathbb{R}$;

(4) u is upper semi-continuous, that is, $\{t : u(t) \ge \alpha\}$ is closed for each $\alpha \in [0, 1]$.

As analyzed in [6,8,9], fuzzy numbers are an extension of intervals, rather than real numbers. So, a fuzzy number is also called a fuzzy interval.

The set of all fuzzy numbers is denoted by $\mathcal{F}$. The set $\mathbb{R}$ of real numbers is canonically embedded in $\mathcal{F}$, identifying each real number a with the "crisp" fuzzy number $\tilde{a} : \mathbb{R} \longrightarrow [0, 1]$ given by

$$\tilde{a}(t) = \begin{cases} 1, & t = a; \\ 0, & \text{otherwise.} \end{cases}$$

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Given a fuzzy number u and $\alpha \in (0, 1]$, the α-level set $u_\alpha = \{t : u(t) \geq \alpha\}$ is a closed interval, hence, $u_\alpha = [u^L(\alpha), u^R(\alpha)]$ for some real numbers $u^L(\alpha)$ and $u^R(\alpha)$. Letting $u^L(0) = \inf\{u^L(\alpha) : 0 < \alpha \leq 1\}$ and $u^R(0) = \sup\{u^R(\alpha) : 0 < \alpha \leq 1\}$, we obtain an increasing function $u^L : [0, 1] \longrightarrow \mathbb{R}$ and a decreasing function $u^R : [0, 1] \longrightarrow \mathbb{R}$. Both u^L and u^R are left continuous on $(0, 1]$, and right continuous at 0. These properties of u^L, u^R lead to the following representation of fuzzy numbers:

Theorem 1.1. *(See [13].) Suppose $f, g : [0, 1] \longrightarrow \mathbb{R}$ satisfy the following conditions:*

(1) *f is increasing and g is decreasing;*
(2) *$f(1) \leq g(1)$;*
(3) *both f and g are left continuous on $(0, 1]$; and*
(4) *both f and g are right continuous at 0.*

Then there is a unique fuzzy number u such that $u^L = f$ and $u^R = g$.

Addition $\oplus$ and multiplication $\otimes$ of fuzzy numbers are defined by Zadeh's extension principle [7]:

$$u \oplus v(t) = \sup\{\min\{u(r),\ v(s)\} : r + s = t\}$$

and

$$u \otimes v(t) = \sup\{\min\{u(r),\ v(s)\} : r \times s = t\}.$$

It is easily verified that for each $\alpha \in [0, 1]$,

$$(u \oplus v)^L(\alpha) = u^L(\alpha) + v^L(\alpha),$$
$$(u \oplus v)^R(\alpha) = u^R(\alpha) + v^R(\alpha),$$
$$(u \otimes v)^L(\alpha) = \min\{u^i(\alpha) \times v^j(\alpha) : i, j \in \{L, R\}\},$$

and

$$(u \otimes v)^R(\alpha) = \max\{u^i(\alpha) \times v^j(\alpha) : i, j \in \{L, R\}\}.$$

Both $(\mathcal{F}, \oplus, \tilde{0})$ and $(\mathcal{F}, \otimes, \tilde{1})$ are commutative monoids. But, the multiplication $\otimes$ does not distribute over the addition $\oplus$ as shown in [25]. This defect of the operations $\oplus, \otimes$ has led to investigations of new arithmetics on fuzzy numbers, see e.g. [7,11,15,17,30].

Representation and calculus of fuzzy numbers have been an interesting topic. Roughly speaking, there are two approaches to the representation of fuzzy numbers. The first is to embed fuzzy numbers into spaces of mathematical entities with which we are familiar. The representation given in Theorem 1.1 is an extremely nice example in this regard. This representation is also related to the idea in [6,8,9] to "represent a fuzzy interval by two fuzzy bounds (gradual numbers) just like a classical interval can be presented as a pair of reals, representing the two bounds of the interval." Other representations in this vein can be found in [21,29,31]. The second approach is to consider some (application-oriented) features of fuzzy numbers. For example, Delgado, Vila, and Voxman [5] associated a *value* and an *ambiguity* with a fuzzy number and obtained a representation of fuzzy numbers. Other representations in this approach can be found in [11,16,30].

A representation of fuzzy numbers might lead to a calculus. For instance, the representation in Theorem 1.1 is the basis to extend arithmetics of intervals to fuzzy numbers, see e.g. [2,3,6,8,9,21,29]. A systematic discussion on arithmetics of intervals and their extensions in the fuzzy setting can be found in the recent [4].

In this paper, a new representation of fuzzy numbers is given. This representation belongs to the first approach, and is related to the additive decomposition of fuzzy numbers discussed in [18,22–24,26]. It is shown that (i) there is a bijection between the set of symmetric fuzzy numbers and the set $D_0[0, 1]$ of decreasing functions $[0, 1] \longrightarrow [0, \infty)$ that are left continuous on $(0, 1]$ and right continuous at 0; (ii) there is a one to one correspondence between the set of skew fuzzy numbers (defined below) and the set $BV_0[0, 1]$ of functions $[0, 1] \longrightarrow \mathbb{R}$ that are left continuous on $(0, 1]$, right continuous at 0, and are of bounded variation; and (iii) each fuzzy number can be written in a unique way as the sum of a symmetric fuzzy number and a skew fuzzy number, with the skew part being its *Mareš core* in the sense