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Characterizations of generalized differentiable fuzzy functions [☆]

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Abstract

This article deals with generalized differentiable fuzzy functions. Specifically, we give some characterizations of generalized Hukuhara differentiable fuzzy functions through the differentiability of their endpoint functions. Then, we introduce a differentiability concept that is more general than the generalized Hukuhara differentiability and extend the above characterization to this new type of differentiability as well as to g-differentiable fuzzy functions. These characterizations are useful tools for the calculus of derivatives of fuzzy functions.

Keywords: Fuzzy functions; Differentiable fuzzy functions; Generalized Hukuhara differentiability; g-Derivative

1. Introduction

A first approach to differentiability for fuzzy-valued functions (fuzzy functions) was given by Puri and Ralescu [21] as a generalization of the Hukuhara derivative for set-valued mappings, which is based on the Hukuhara difference of sets [13]. With the concept of Hukuhara differentiable fuzzy function (H-differentiable, for short), fuzzy differential equations emerged [15,16,20,22,23].

The H-differentiability of fuzzy functions is a rather restrictive concept of derivative. In fact, if a fuzzy function F is H-differentiable, then it possesses the property that the diameter of the support, $\mathrm{len}(F_0(t))$ (length of $F_0(t)$), is nondecreasing as t increases. Thus, a simple fuzzy function such as $F(t) = u \cdot t$, where u is a fuzzy interval and $\cdot$ denotes the multiplication of u by a real number t, is not an H-differentiable fuzzy function. To overcome this difficulty, the authors of reference [2] introduced the concept of strongly generalized differentiable fuzzy function

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(G-differentiable, for short) by considering one-sided H-derivatives. The class of G-differentiable fuzzy functions is more general than that of H-differentiable fuzzy functions, in the sense that the latter is strictly contained in the former. This differentiability concept coincides with the one introduced in [14].

The concept of G-differentiable fuzzy function is widely used in fuzzy differential equations, obtaining new solutions and showing the existence of periodic solutions [1,3,5,6,9,14,17,18,24,27]. This class of functions is also of major importance in fuzzy optimization. In fact, G-differentiable fuzzy functions have been used to define generalized convex fuzzy functions and to obtain necessary and sufficient conditions for the existence of local and global minima [8,10].

Nevertheless, there are some fuzzy functions having the form $F(t) = u \cdot g(t)$, where u is a fuzzy interval and $g : [a, b] \to \mathbb{R}$ is a differentiable function, that are not G-differentiable. This restriction is due to the use of the H-difference in the definition of the G-derivative. A more general concept of differentiability is obtained if a less restrictive difference definition is used. The generalized Hukuhara difference has been recently introduced in [26], being more general than the H-difference. With this new concept of difference between fuzzy intervals, the generalized Hukuhara differentiable fuzzy functions (gH-differentiable, for short) are studied in [4]. In the same article [4], the concepts of g-difference between fuzzy functions and g-differentiable fuzzy functions are also introduced. Sufficient conditions for the gH-differentiability and the g-differentiability of fuzzy functions are also given in the cited paper, where the authors also show by means of an example that such conditions are not necessary.

In this paper, we give a characterization of gH-differentiable fuzzy functions. This characterization is obtained via the differentiability of the lateral or endpoint functions, providing a useful tool to obtain the gH-derivative of any fuzzy function. Then we introduce a new differentiability concept, which enlarges the class of fuzzy differentiable functions. We also characterize this new class of differentiable functions in terms of the differentiability of the endpoint functions. This characterization is extended to the case of g-differentiable fuzzy functions. Several examples illustrate the obtained result.

2. Notation and basic results

We denote by $\mathcal{K}_C$ the family of all bounded closed intervals in $\mathbb{R}$, i.e.,

$$\mathcal{K}_C = \left\{ A = \left[\underline{a}, \overline{a}\right] \;/\; \underline{a}, \overline{a} \in \mathbb{R} \text{ and } \underline{a} \leq \overline{a} \right\}.$$

Definition 1. (See [25].) The generalized Hukuhara difference of two intervals $A, B \in \mathcal{K}_C$ (gH-difference, for short), denoted by $A \ominus_{gH} B$, is defined by

$$A \ominus_{gH} B = C \Leftrightarrow \begin{cases} & \text{(i) } A = B + C, \\ \text{or} & \text{(ii) } B = A + (-1)C. \end{cases}$$

The following properties have been obtained in [25,26].

Proposition 1. *The gH-difference $\ominus_{gH}$ on $\mathcal{K}_C$ has the following properties:*

(1) *the gH-difference $A \ominus_{gH} B$ always exists for all $A, B \in \mathcal{K}_C$.*
(2) *$A \ominus_{gH} A = \{0\}$ for all $A \in \mathcal{K}_C$.*
(3) *given $A, B \in \mathcal{K}_C$, either $A + (B \ominus_{gH} A) = B$ or $B + (-1)(B \ominus_{gH} A) = A$ and both equalities hold if and only if $B \ominus_{gH} A$ is a singleton set.*

Given two intervals $A = [\underline{a}, \overline{a}]$ and $B = [\underline{b}, \overline{b}]$, we define the distance between A and B by

$$H(A, B) = \max\left\{\left|\underline{a} - \underline{b}\right|, \left|\overline{a} - \overline{b}\right|\right\}.$$

A fuzzy set on $\mathbb{R}$ is a mapping $u : \mathbb{R} \to [0, 1]$. For each fuzzy set u, we denote by $[u]^\alpha = \{x \in \mathbb{R} : u(x) \geq \alpha\}$, for any $\alpha \in (0, 1]$, its α-level set. By $supp(u)$ we denote the support of u, i.e., $\{x \in \mathbb{R} : u(x) > 0\}$. By $[u]^0$ we represent the closure of $supp(u)$, i.e., $[u]^0 = cl(supp(u))$, where $cl(M)$ denotes the closure of the subset $M \subseteq \mathbb{R}$.

Definition 2. A fuzzy set u on $\mathbb{R}$ is said to be a fuzzy interval if: