Available online at www.sciencedirect.com

ScienceDirect

Fuzzy Sets and Systems 295 (2016) 57–71

FUZZY sets and systems

www.elsevier.com/locate/fss

Relationship between Bede–Gal differentiable set-valued functions and their associated support functions

Şahin Emrah Amrahov [a,*], Alireza Khastan [b], Nizami Gasilov [c], Afet Golayoglu Fatullayev [c]

[a] *Computer Engineering Department, Ankara University, 06830 Turkey*
[b] *Department of Mathematics, Institute for Advanced Studies in Basic Sciences (IASBS), Zanjan 45137-66731, Iran*
[c] *Baskent University, Ankara, 06810 Turkey*

Received 24 October 2014; received in revised form 26 November 2015; accepted 4 December 2015

Available online 17 December 2015

Abstract

In this study, we adapt the concept of the Bede–Gal derivative, which was initially suggested for fuzzy number-valued functions, to set-valued functions. We use an example to demonstrate that this concept overcomes some of the shortcomings of the Hukuhara derivative.

We prove some properties of Bede–Gal differentiable set-valued functions. We also study the relationship between a Bede–Gal differentiable set-valued function and its value's support function, which we call the associated support function. We provide examples of set-valued functions that are not Bede–Gal differentiable whereas their associated support functions are differentiable. We also present some applications of the Bede–Gal derivative to solving set-valued differential equations.

Keywords: Bede–Gal differentiability; Set-valued differential equation; Set-valued function; Support function

1. Introduction

Recently, many studies have examined the properties of set-valued functions [15,34,42]. In fact, this is the second time in mathematical history. Previously, at the beginning of the 1960s, set-valued functions were explored to study the properties of controllable dynamic systems. Reformulating differential equations that describe the behavior of controllable systems in the form of differential inclusion is a very important idea. Indeed, the notion of differential inclusion was actually expressed in earlier studies during the 1930s, e.g., Marchaud [33] and Zaremba [45]. However, these studies yielded no applications and thus they did not attract the attention of other researchers. In the 1960s, proving the Pontryagin maximum principle for optimal control [37] inspired further studies of the properties of set-valued functions and many important results were reported [5,6,8,11,13,14,17,43]. The proposal of fuzzy logic and fuzzy sets

* Corresponding author. Tel.: +90 312 2033300x1771.
E-mail address: emrah@eng.ankara.edu.tr (Ş.E. Amrahov).

http://dx.doi.org/10.1016/j.fss.2015.12.002

by Zadeh [44] in 1965 also failed to attract the attention of researchers for 10 years. However, the theory of fuzzy sets was applied subsequently in many areas. A definition for the derivative of fuzzy-valued functions was necessary for the further development of this theory and several definitions have been provided [23,40]. In 2005, Bede and Gal provided a definition of the generalized derivative for fuzzy number-valued functions [9], and this definition of the fuzzy derivative now has wide acceptance among researchers. In this study, we refer to the generalized derivative as the Bede–Gal derivative. In fact, the development of the theory of fuzzy differential equations accelerated following this definition [1–3,25,26,29,35,39]. Approaches are also available that do not employ the Bede–Gal derivative [4,7, 18–22,24] due to some limitations of this concept. Therefore, some studies have aimed to generalize the Bede–Gal derivative [10,41].

Basically, the fuzzy number is defined as a special set. Recent studies have tried to build a bridge between set-valued functions and fuzzy number-valued functions. In some studies, results related to fuzzy sets theory have been adapted to set-valued or interval-valued functions. For example, Malinowski [28,30–32] used a second type Hukuhara derivative to solve interval-valued and set-valued differential equations. Stefanini and Bede [42] introduced the Bede–Gal derivative for interval-valued functions. In [15,16], some properties of Bede–Gal differentiable interval-valued functions were studied.

Similarly, a concept of the derivative for set-valued functions can facilitate the study of set-valued differential equations. However, to solve a set-valued differential equation without introducing this concept, we must use either the Hukuhara derivative [23] or an approach similar to Hüllermeier's approach [24] for fuzzy differential equations. In Hüllermeier's approach, a set-valued differential equation is interpreted based on differential inclusions. This approach is a powerful tool for theoretical investigations of set-valued differential equations, but calculating the solution according to this method is a complex task. This approach has the following characteristics. 1) There is no concept of the derivative for set-valued functions, which needs to be addressed; therefore, we cannot develop an antiderivative (integral) to solve differential equations in an effective manner. 2) A solution in the form of a set-valued function that satisfies the given conditions, such as compactness and convexity, might not always exist. Thus, even if we obtain a solution (or a candidate for the solution is given), we need to prove that it is an appropriate solution, i.e., that it satisfies the compactness and convexity conditions.

In this study, we adapt the Bede–Gal derivative to a family of functions with values that are nonempty compact subsets of $\mathbb{R}^n$, which allows us to avoid the main difficulties when implementing the differential inclusion approach. We illustrate this statement by a simple example. Let us consider the problem of finding the solution to the initial value problem (IVP): $x'(t) = [-1, 1]$, $x(0) = 0$. If we do not have an appropriate concept for the differentiability of set-valued function, we may treat this problem as a differential inclusion problem $x'(t) \in [-1, 1]$, $x(0) = 0$. We can see that each of the functions $x(t) = 0$, $x(t) = \sin t$, $x(t) = \cos t - 1$, etc., is a solution. Moreover, it can be shown that there is an infinite number of solutions. Now, consider the set of all values of these solutions at a certain point t as the value $X(t)$ ($X(t) \subseteq \mathbb{R}$) for the solution X. To study the properties of this solution, we need to determine the set $X(t)$, which is not an easy task in the general case. However, we can see that the aforementioned IVP has the solution $x(t) = [-t, t]$ in the sense of the Bede–Gal derivative for set-valued (interval-valued) functions (see Section 5). We can extend this example to the case of $\mathbb{R}^n$. In particular, we can consider the set-valued IVP $X' = B$ and $X(0) = \{0\}$ instead of the differential inclusion problem $x'(t) \in B$ and $x(0) = 0$, where $B \subseteq \mathbb{R}^n$ is the closed unit ball centered at the origin. We can show that the aforementioned set-valued IVP has the solution $X(t) = t\,B$ (see Section 5).

Similar to [9], we may construct an example to illustrate the drawback of the Hukuhara derivative for set-valued functions. If $g : (a, b) \to \mathbb{R}$ is differentiable at $t_0 \in (a, b)$ with $g'(t_0) < 0$, then the function $F(t) = g(t)\,A$ is not Hukuhara differentiable at t_0, where A is a compact and convex set in $\mathbb{R}^n$. In Section 3, we show that $F(t)$ is Bede–Gal differentiable and we have $F'(t) = g'(t)\,A$. Indeed, the Bede–Gal derivative is defined for a larger class of set-valued functions compared with the Hukuhara derivative.

The remainder of this paper is organized as follows. In Section 2, we provide some preliminary information. In Section 3, we introduce the Bede–Gal derivative for set-valued functions. We also prove some properties of Bede–Gal differentiable set-valued functions. In Section 4, we study the relationships between Bede–Gal differentiable set-valued functions and their associated support functions. We provide some examples of set-valued functions that are not Bede–Gal differentiable whereas their associated support functions are differentiable. In Section 5, we apply our results to solve some set-valued differential equations. In Section 6, we give our concluding remarks.