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Ill-posed fuzzy initial-boundary value problems based on generalized differentiability and regularization ☆

Zengtai Gong [a,*], Hong Yang [a,b]

[a] College of Mathematics and Statistics, Northwest Normal University, Lanzhou 730070, PR China
[b] College of Technology and Engineering, Lanzhou University of Technology, Lanzhou 730050, PR China

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Abstract

In this study, based on a new generalized differentiability and parametric representation for fuzzy valued functions, we discuss the existence of solutions for fuzzy partial differential equations (FPDEs) and their ill-posedness. A regularization method is required to recover the numerical stability. A stable approximation of the exact solution is obtained as an example of a Cauchy problem for the fuzzy Laplace equation.

Keywords: Fuzzy partial; Fuzzy partial differential equation; Generalized differentiability; Ill-posedness; Regularization method

1. Introduction

When physical phenomena and various problems in social science are transformed into deterministic problems based on partial differential equations (PDEs), it is generally the case that the model is imperfect. The study of fuzzy partial differential equations (FPDEs) provides a suitable setting for the mathematical modeling of real-world problems that include uncertainty or vagueness. Thus, ordinary differential equations are replaced by FPDEs, which are more appropriate tools for modeling with uncertainty.

As a new and powerful mathematical tool, FPDEs have been studied using several approaches. The first definition of an FPDE was presented by Buckley and Feuring in [5]. Allahviranloo [2] proposed a difference method for solving FPDEs. The Adomian decomposition method was studied for finding the approximate solution of the fuzzy heat equation in [1]. Solving FPDEs by the differential transformation method was considered in [3]. In [18], the authors considered the application of FPDEs obtained using fuzzy rule-based systems. Furthermore, Oberguggenberger described weak and fuzzy solutions for FPDEs [26] and Chen et al. presented a new inference method with applications to FPDEs [9]. In [15], an interpretation was provided of the use of FPDEs for modeling hy-

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* Corresponding author. Tel.: +86 931 7971845.
E-mail addresses: gongzt@nwnu.edu.cn, zt-gong@163.com (Z. Gong), yhong1978@163.com (H. Yang).

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drogeological systems. Studies of heat, wave, and Poisson equations with uncertain parameters were provided in [8]. Recently, fuzzy solutions for heat equations based on generalized Hukuhara differentiability were considered in [4].

The concept of fuzzy sets was originally introduced by Zadeh [37] and it led to the definition of the fuzzy number, as well as its application to fuzzy control [10] and approximate reasoning problems [38,39]. The fuzzy mapping function was introduced by Chang and Zadeh [10]. Later, Dubois and Prade [11] presented an elementary fuzzy calculus based on the extension principle [37]. Puri and Ralescu [28] suggested two definitions for the fuzzy derivative of fuzzy functions. The first method was based on the H-difference notation and it was investigated further by Kaleva [19]. The second method was derived from the embedding technique, where Goetschel and Voxman [16] gave a more applicable representation. It is well known that the usual Hukuhara difference between two fuzzy numbers exists only under very restrictive conditions [11,28,19]. The gH-difference of two fuzzy numbers exists under much less restrictive conditions, but it does not always exist [31,30]. The g-difference proposed in [32] overcomes the shortcomings of the concepts mentioned above and the g-difference of two fuzzy numbers always exists under much less restrictive conditions. The same is valid if we consider differentiability concepts in fuzzy settings. Based on the gH-difference described by [32,33], new gH-derivative concepts that generalize those in [6] have been investigated, mainly in terms of their characterization. Using the g-difference a new, very general fuzzy differentiability concept was defined and studied, which is called the g-derivative in [7].

The present study addresses two issues. First, we consider the ill-posedness of FPDEs, where we discuss the generalized partial differentiation of fuzzy valued functions to ensure the existence of solutions for FPDEs. Second, we use the regularization method to recover the numerical stability. In 1923, Hadamard [17] introduced the concept of a well-posed problem from philosophy where the mathematical model of a physical problem must have properties where the solution exhibits uniqueness, existence, and stability. If one of the properties fails to hold, the problem is known as ill-posed. Numerical computation is difficult due to the ill-posedness of the problem, so we define the ill-posedness of FPDEs using the decomposition theorem and we propose a regularization method for computing their approximate solutions. These problems occur in many real-world applications, such as nondestructive testing techniques [20,22,23], geophysics [34], and cardiology [14].

The remainder of this paper is organized as follows. In Section 2, we briefly introduce the necessary notions related to fuzzy numbers. In Section 3, we introduce the new concept of a generalized partial derivative that ensures the existence of a solution for FPDEs. In Section 4, we define the solution and ill-posedness of FPDEs. The regularization method and convergence estimates for the initial-boundary value problems of FPDEs are considered in Section 5. In Section 6, we present some numerical results and our conclusions are given in Section 7.

2. Preliminaries

Let $P_k(R^n)$ denote the family of all the nonempty compact convex subsets of R^n and define the addition and scalar multiplication in $P_k(R^n)$ in the usual manner. Let A and B be two nonempty bounded subsets of R^n. The distance between A and B is defined by the Hausdorff metric

$$d_H(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B}||a-b||, \sup_{b\in B}\inf_{a\in A}||b-a||\right\}, \tag{2.1}$$

where $||\cdot||$ denotes the usual Euclidean norm in R^n [12]. Then, $(P_k(R^n); d_H)$ is a metric space.

Denote

$$E^n = \{u : R^n \to [0,1] | u \text{ satisfies (1)–(4) below}\}$$

as a fuzzy number space where

(1) u is normal, i.e. there exists an $x_0 \in R^n$ such that $u(x_0) = 1$,
(2) u is fuzzy convex, i.e., $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$ for any $x, y \in R^n$ and $0 \leq \lambda \leq 1$,
(3) u is upper semi-continuous,
(4) $[u]_0 = cl\{x \in R^n | u(x) > 0\}$ is compact.