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Fuzzy Sets and Systems 261 (2015) 1–19

FUZZY sets and systems

www.elsevier.com/locate/fss

The general nilpotent operator system

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Received 30 August 2013; received in revised form 4 May 2014; accepted 6 May 2014

Available online 21 May 2014

Abstract

In this paper we show that a consistent logical system generated by nilpotent operators is not necessarily isomorphic to Łukasiewicz-logic, which means that nilpotent logical systems are wider than we have thought earlier. Using more than one generator functions we examine three naturally derived negations in these systems. It is shown that the coincidence of the three negations leads back to a system which is isomorphic to Łukasiewicz-logic. Consistent nilpotent logical structures with three different negations are also provided.

Keywords: t-norm; t-conorm; Nilpotent operator; Lukasiewicz operator system; Negation operator; De Morgan class

1. Introduction

One of the most significant problems of fuzzy set theory is the proper choice of set-theoretic operations [22,26]. Triangular norms and conorms have thoroughly been examined in the literature [15,10,9,12]. The most well-characterized class of t-norms are the so-called representable t-norms. They are derived from the solution of the associative functional equation [1]. The two main types of representable t-norms are the strict and non-strict or nilpotent t-norms.

t-norms and t-conorms are often used as conjunctions and disjunctions in logical structures [11,18]. Łukasiewicz fuzzy logic [13,17,19,20] is the logic where the conjunction is the Łukasiewicz t-norm. It has been introduced for philosophical reasons by Łukasiewicz in [17] and it is among the most significant and widely examined non-classical logics.

The class of non-strict t-norms has preferable properties which make them more usable in building up logical structures. Among these properties are the fulfillment of the law of contradiction and the excluded middle, the continuity of the implication or the coincidence of the residual and the S-implication [8,25]. Due to the fact that all continuous Archimedean (i.e. representable) nilpotent t-norms are isomorphic to the Łukasiewicz t-norm [12], the previously studied nilpotent systems were all isomorphic to the well-known Łukasiewicz-logic.

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http://dx.doi.org/10.1016/j.fss.2014.05.011

In this paper we show that a logical system generated by nilpotent operators is not necessarily isomorphic to Łukasiewicz-logic. Of course, this lack of isomorphy is not the result of introducing a new operator family, it simply means that the system itself is built up in a significantly different way using more than one generator functions.

The paper is organized as follows. After some preliminaries in Section 2 we give a characterization of negation operators in Section 3, as negations will have an important role to play in Section 4. After considering the class of connective systems generated by nilpotent operators, we examine their structural properties in Section 4. We show examples for bounded systems, i.e. consistent nilpotent systems which are not isomorphic to Łukasiewicz-logic. Necessary and sufficient conditions are given for these systems to satisfy the De Morgan law, classification property and consistency. A wide range of examples for consistent and non-consistent bounded systems can be found in Section 5.

2. Preliminaries – basic fuzzy connectives

2.1. t-Norms and t-conorms

First, we recall some basic notations and results regarding t-norms, t-conorms and negation operators that will be useful in the sequel.

A triangular norm (*t-norm* for short) T is a binary operation on the closed unit interval $[0, 1]$ such that $([0, 1], T)$ is an abelian semigroup with neutral element 1 which is totally ordered, i.e., for all $x_1, x_2, y_1, y_2 \in [0, 1]$ with $x_1 \leq x_2$ and $y_1 \leq y_2$ we have $T(x_1, y_1) \leq T(x_2, y_2)$, where $\leq$ is the natural order on $[0, 1]$.

A triangular conorm (*t-conorm* for short) S is a binary operation on the closed unit interval $[0, 1]$ such that $([0, 1], S)$ is an abelian semigroup with neutral element 0 which is totally ordered.

Standard examples [5,15] of t-norms are the minimum $T_{\mathbf{M}}$, the product $T_{\mathbf{P}}$, the Łukasiewicz t-norm $T_{\mathbf{L}}$ given by $T_{\mathbf{L}}(x, y) = \max(x + y - 1, 0)$, and the drastic product $T_{\mathbf{D}}$ with $T_{\mathbf{D}}(1, x) = T_{\mathbf{D}}(x, 1) = x$, and $T_{\mathbf{D}}(x, y) = 0$ otherwise.

Standard examples of t-conorms are the maximum $S_{\mathbf{M}}$, the probabilistic sum $S_{\mathbf{P}}$, the Łukasiewicz t-conorm $S_{\mathbf{L}}$ given by $S_{\mathbf{L}}(x, y) = \min(x + y, 1)$, and the drastic sum $S_{\mathbf{D}}$ with $S_{\mathbf{D}}(0, x) = S_{\mathbf{D}}(x, 0) = x$, and $S_{\mathbf{D}}(x, y) = 1$ otherwise.

A continuous t-norm T is said to be *Archimedean* if $T(x, x) < x$ holds for all $x \in (0, 1)$, *strict* if T is strictly monotone i.e. $T(x, y) < T(x, z)$ whenever $x \in (0, 1]$ and $y < z$, and *nilpotent* if there exist $x, y \in (0, 1)$ such that $T(x, y) = 0$.

From the duality between t-norms and t-conorms we can easily get the following properties as well. A continuous t-conorm S is said to be *Archimedean* if $S(x, x) > x$ holds for every $x, y \in (0, 1)$, *strict* if S is strictly monotone i.e. $S(x, y) < S(x, z)$ whenever $x \in [0, 1)$ and $y < z$, and *nilpotent* if there exist $x, y \in (0, 1)$ such that $S(x, y) = 1$.

Proposition 1. *(See [16,4].) A function* $T : [0, 1]^2 \to [0, 1]$ *is a continuous Archimedean t-norm iff it has a continuous additive generator, i.e. there exists a continuous strictly decreasing function* $t : [0, 1] \to [0, \infty]$ *with* $t(1) = 0$, *which is uniquely determined up to a positive multiplicative constant, such that*

$$T(x, y) = t^{-1}\big(\min\big(t(x) + t(y), t(0)\big)\big), \quad x, y \in [0, 1]. \tag{1}$$

Proposition 2. *(See [16,4].) A function* $S : [0, 1]^2 \to [0, 1]$ *is a continuous Archimedean t-conorm iff it has a continuous additive generator, i.e. there exists a continuous strictly increasing function* $s : [0, 1] \to [0, \infty]$ *with* $s(0) = 0$, *which is uniquely determined up to a positive multiplicative constant, such that*

$$S(x, y) = s^{-1} \min\big(\min\big(s(x) + s(y), s(1)\big)\big), \quad x, y \in [0, 1]. \tag{2}$$

Proposition 3. *(See [12].) A t-norm* T *is strict if and only if* $t(0) = \infty$ *holds for each continuous additive generator* t *of* T. *A t-norm* T *is nilpotent if and only if* $t(0) < \infty$ *holds for each continuous additive generator* t *of* T. *A t-conorm* S *is strict if and only if* $s(1) = \infty$ *holds for each continuous additive generator* s *of* S. *A t-conorm* S *is nilpotent if and only if* $s(1) < \infty$ *holds for each continuous additive generator* s *of* S.

In both of the above mentioned Propositions 1 and 2 we can allow the generator functions to be strictly increasing *or* strictly decreasing, which will result in the fact that they will be determined up to a (not necessarily positive) multiplicative constant. For an increasing generator function t of a t-conorm and similarly for a decreasing generator