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Superdecomposition integrals

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Abstract

This study introduces and discusses a new class of integrals based on superdecompositions of integrated functions, including an analysis of their relationship with decomposition integrals, which were introduced recently by Even and Lehrer. The proposed superdecomposition integrals have several properties that are similar or dual with respect to decomposition integrals, but they also have some significant differences. The convex integral is obtained by considering all possible superdecompositions with no constraints on the applied sets, which can be treated as the greatest convex homogeneous functional that is bounded from above by the measure we consider. The relationship with the universal integral of Klement et al. is also discussed. Finally, some possible generalizations are outlined.

Keywords: Choquet integral; Convex integral; Decomposition integral; Monotone measure; Superdecomposition integral

1. Introduction

Integrals play a key role in many theoretical and applied areas where the information contained in a measure (for example, weights of groups of criteria) and a function (for example, a score vector) is expressed using a single representative value. The origin of integrals is linked to measuring actual descriptions of the physical world, such as the length, area, and volume, which are sigma-additive, and real-valued functions. More general measures were considered only in the last century, especially their associations with human sciences. For example, the interactions within groups of people cannot be modeled directly using additive measures. Modified techniques were also introduced for constructing classical integrals such as the Riemann integral in 1854 and the Lebesgue integral in 1902. Various approaches developed for general integration include the Choquet integral [2] and the Sugeno integral [20]. For state-of-the-art accounts of generalized measures and integral theory (real-valued functions and monotone

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measures), we recommend a published handbook [16] and previous monographs [3,4,6,15,23] and [1,10,11,17,24,25]. Recently, Lehrer and Teper [8,9,22] introduced a concave integral, which can be treated as a solution to an optimization problem that maximizes the lower integral sums, i.e., it is based on a subdecomposition of a considered function. A common framework for Lehrer and Teper's concave integral and the Choquet integral was proposed by Even and Lehrer [5], who introduced the decomposition integrals. These decomposition integrals maximize the lower integral sums related to subdecompositions of the functions under consideration given some constraints on the sets being considered. Klement et al. [7] introduced a framework for functionals based on monotone measures, which should be referred to as universal integrals. Integrals that are both universal and decomposition integrals were characterized by Stupňanová [19].

Inspired by the idea of decomposition integrals, we introduce and study a dual view of integration based on the upper integral sums, i.e., superdecompositions of the functions being considered. In the next section, we introduce decomposition and universal integrals, as well as some results related to these special integrals. Section 3 describes the proposed superdecomposition integrals, including several examples. In Section 4, we discuss the convex integral as a special superdecomposition integral related to convex functionals. Finally, some concluding remarks are provided, including possible further generalizations obtained by modifying the arithmetical operations applied.

2. Universal and decomposition integrals

2.1. Universal integral

Let $\mathcal{A}$ be a σ-algebra of subsets of set X. A set function $m : \mathcal{A} \to [0, \infty]$ is called a *monotone measure* whenever $m(\emptyset) = 0 < m(X)$, and for every $A, B \in \mathcal{A}$ such that $A \subseteq B$ we have $m(A) \le m(B)$. The following concepts are needed to define a universal integral.

Definition 1. (See [7].) Let $(X, \mathcal{A})$ be a measurable space.

(i) $\mathcal{F}^{(X,\mathcal{A})}$ is the set of all $\mathcal{A}$-measurable functions $f : X \to [0, \infty]$.
(ii) For each number $a \in]0, \infty]$, $\mathcal{M}_a^{(X,\mathcal{A})}$ is the set of all monotone measures that satisfy $m(X) = a$, and we take

$$\mathcal{M}^{(X,\mathcal{A})} = \bigcup_{a \in]0,\infty]} \mathcal{M}_a^{(X,\mathcal{A})}.$$

An equivalence relation between pairs of measures and functions was introduced in [7].

Definition 2. Two pairs, $(m_1, f_1) \in \mathcal{M}^{(X_1,\mathcal{A}_1)} \times \mathcal{F}^{(X_1,\mathcal{A}_1)}$ and $(m_2, f_2) \in \mathcal{M}^{(X_2,\mathcal{A}_2)} \times \mathcal{F}^{(X_2,\mathcal{A}_2)}$, which satisfy

$$m_1\big(\{f_1 \ge t\}\big) = m_2\big(\{f_2 \ge t\}\big) \text{ for all } t \in]0, \infty],$$

are called integral equivalent, which are represented as

$$(m_1, f_1) \sim (m_2, f_2).$$

Integral equivalence can be viewed as a generalization of the stochastic equivalence of random variables. Thus, two random variables (possibly defined on two different probability spaces) are integral equivalent, i.e., $(X, P_1) \sim (Y, P_2)$ if and only if they have coincident distribution functions, $F_X = F_Y$.

The notion of pseudo-multiplication is required to describe the universal integral.

Definition 3. (See [15,21].) A function $\otimes : [0, \infty]^2 \to [0, \infty]$ is called a pseudo-multiplication if it satisfies the following properties:

(i) It is non-decreasing in each component, i.e., for all $a_1, a_2, b_1, b_2 \in [0, \infty]$ with $a_1 \le a_2$ and $b_1 \le b_2$, we have $a_1 \otimes b_1 \le a_2 \otimes b_2$;
(ii) 0 is an annihilator of $\otimes$, i.e., for all $a \in [0, \infty]$, we have $a \otimes 0 = 0 \otimes a = 0$;