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Fuzzy Sets and Systems 259 (2015) 56–67

FUZZY sets and systems

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States on quantum and algebraic structures and their integral representation ☆

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Available online 11 November 2013

Abstract

We review the notion of a state as a finitely additive functional on algebraic and quantum structures such as MV-algebras, effect algebras, and BL-algebras and their non-commutative generalizations. We show how a state can be represented as a standard integral through a σ-additive Borel probability measure.
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Keywords: Effect algebra; Pseudo effect algebra; MV-algebra; Pseudo MV-algebra; State; ℓ-Group; Unital ℓ-group; Strong unit; Riesz decomposition property; Pseudo BL-algebra; Integral representation

1. Introduction

It has been 80 years since Kolmogorov published a book on the foundations of probability theory [28]. This was the first time that probability theory was axiomatized and presented as a rigorous mathematical branch and not just as a mathematical toy for solving puzzles inspired by hazard games. According to Kolmogorov, a probability measure is a σ-additive probability measure P defined on a σ-algebra $\mathcal{S}$ of subsets of a set $\Omega \neq \emptyset$. This model became very important and is still used in basic courses on probability theory in universities. However, immediately after publication, it was recognized that the Kolmogorov axioms cannot describe all the measurements in new physics of the last century that we now call quantum mechanics.

The Heisenberg uncertainty principle states that the position x and momentum p of an elementary particle cannot be measured simultaneously with arbitrarily prescribed accuracy. If $\Delta_m p$ and $\Delta_m x$ denote the inaccuracy for measurement of p and x in state m, then

$$(\Delta_m p)^2 \cdot (\Delta_m x)^2 \geqslant \frac{1}{4}\hbar^2 > 0, \tag{1.1}$$

where $\hbar = h/2\pi$ and h is Planck's constant. Birkhoff and von Neumann showed that quantum mechanical events satisfy more general axioms than Boolean algebras do [3], which we call quantum logic or quantum structures. Today

☆ This work was supported by the Slovak Research and Development Agency under contract APVV-0178-11, grant VEGA No. 2/0059/12 SAV, and CZ.1.07/2.3.00/20.0051.

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http://dx.doi.org/10.1016/j.fss.2013.11.005

we have a whole hierarchy of quantum structures such as Boolean algebras, orthomodular lattices and posets, orthoalgebras, D-posets, and effect algebras [17]. Quantum structures are algebraic structures for which the basic operations are often partial.

D-posets were introduced by Kôpka and Chovanec [29], for which the basic notion is the difference of two comparable events. An equivalent structure is an effect algebra [21], for which the primary notion is addition $+$. For the partial operation $+$, $a+b$ means the disjunction of two mutually exclusive events a and b. A prototypical example of effect algebras is the system $\mathcal{E}(H)$ of all Hermitian operators of a real, complex, or quaternionic Hilbert space H that are between the zero operator, O, and the identity operator, I. This models so-called POV measures, which is a basic tool for quantum measurement in Hilbert space quantum mechanics.

An analogue of a probability measure of an effect algebra E is the notion of a state, which is a finitely additive mapping $s: E \to [0,1]$ that preserves all existing sums $a+b$ and is normalized, that is, $s(1)=1$. This notion corresponds to the notion of a probability measure in the sense of de Finetti [5,6], who assumed that a probability measure has to be only finitely additive.

Another important example of effect algebras is an Abelian po-group (partially ordered group) G with a fixed positive element u. If we restrict our considerations to the interval $\Gamma(G,u) := [0,u] = \{g \in G\colon\ 0 \leqslant g \leqslant u\}$, we obtain an effect algebra with the group addition restricted to $[0,u]$. Such an effect algebra is said to be an interval effect algebra. It can happen that an effect algebra is stateless; however, every interval effect algebra has at least one state [26, Corollary 4.4]. An important property that guarantees that an effect algebra E is an interval effect algebra is the Riesz decomposition property (RDP), according to which, every two decompositions of a unit element have a joint refinement [36]. Then E is isomorphic to an interval $[0,u]$ in some Abelian po-group G with strong unit u.

In 1958, Chang introduced MV-algebras for modeling infinitely valued Łukasziewicz logic [4]. According to the well-known Mundici result, every MV-algebra is an interval in some unital Abelian ℓ-group with a strong unit, and there is a categorical equivalence between the variety of MV-algebras and the category of unital Abelian ℓ-groups [31].

It was immediately shown that every MV-algebra M is in fact an effect algebra [29]. In addition, M is a lattice-ordered effect algebra satisfying RDP, and conversely, every lattice-ordered effect algebra satisfying RDP can be viewed as an MV-algebra.

Mundici defined the notion of a state for MV-algebras as an additive functional preserving a partial addition (which corresponds to the group addition) almost 40 years after MV-algebras were introduced [32]. The notion of a state is not a proper one for universal algebra; however, for the theory of quantum structures it is a basic notion, and for D-posets (effect algebras) it was already defined [29].

As already mentioned, there are two concepts of a probability measure: Kolmogorov's approach as a σ-additive measure and de Finetti's concept of a finitely additive measure. The first has a technical advantage, while the second is more intuitive. Kroupa [30] and Panti [33] showed that every state on an MV-algebra can be expressed as a standard integral through a unique regular (σ-additive) Borel probability measure on some Borel σ-algebra of subsets of a compact Hausdorff topological space. This result was also generalized for effect algebras satisfying RDP [11]. As a conclusion of this research on states we can say that there is practically no real difference between the Kolmogorov and de Finetti approaches to a probability measure.

It is important to note that although the original inspiration for the study of quantum structures was quantum mechanics, phenomena such as inequality (1.1) can be observed in different fields such as computer science, psychiatry, neuroscience (quantum brains [39], quantum psychology [40]), and quantum computing.

There are many other new algebraic structures that can be studied in the framework of quantum structures that model uncertainty such as BL-algebras [27], MTL-algebras and their non-commutative generalizations, pseudo MV-algebras [24,37], pseudo effect algebras [19,20], and pseudo BL-algebras [8,9]. Therefore, it is important to introduce an analogue of a state for these structures. For pseudo MV-algebras this is straightforward because according to the basic result that generalizes the representation of MV-algebras [11], any pseudo MV-algebra is an interval in a unital ℓ-group G (not necessarily Abelian) with a strong unit. Hence, we can derive a partial operation $+$ that is in fact the restriction of group addition to the interval $[0,u]$. In other structures, such as BL-algebras and pseudo BL-algebras, it is not clear immediately how we can define a state. Therefore, there are two notions of states for BL-algebras and pseudo BL-algebras: Bosbach states [23] and Riečan states [38].

Here we present an overview of states on some quantum and algebraic structures and show when states on different quantum structures can have an integral representation using σ-additive Borel regular measures on appropriate compact Hausdorff spaces.

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