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Weak convergence of nonadditive measures based on nonlinear integral functionals ☆

Jun Kawabe [1]

Faculty of Engineering, Shinshu University, 4-17-1 Wakasato, Nagano 380-8553, Japan

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Abstract

In this paper, we formulate a general portmanteau theorem for a perturbative nonlinear integral functional and discuss the uniformity of weak convergence of nonadditive measures based on such a functional. As their direct consequences, it turns out that Lévy convergence of nonadditive measures coincides with every one of three types of weak convergence, that is, weak Choquet, weak Sugeno, and weak Shilkret convergence and they are uniform on every bounded subset of Lipschitz functions. Those results are applied when discussing the metrizability of the Lévy topology on the space of nonadditive measures and defining the Fortet–Mourier type metrics on a uniformly equi-autocontinuous set of nonadditive measures.

Keywords: Nonadditive measure; Weak convergence of measures; Choquet integral; Sugeno integral; Shilkret integral; Integral functional; Perturbation; Portmanteau theorem

1. Introduction

A nonadditive measure is a monotone set function vanishing at the empty set. This type of set function, in conjunction with several types of nonlinear integrals, has been extensively studied with applications to decision theory under uncertainty, game theory, data mining, some economic topics under Knightian uncertainty and others [5,10,18–20,29,34,35].

When discussing limit theorems of random variables and stochastic processes in probability theory and statistics, we often have a benefit of weak convergence of measures. It gives a topology on the space of σ-additive measures on a topological space, based on the convergence of their Lebesgue integrals with continuous, bounded, real-valued integrands. This kind of convergence is also interesting from a topological measure theoretic view, since it is closely related to the topology of the space on which the measures are defined. See [1,3,21,31] for more account of weak convergence of measures, with applications to probability theory and statistics.

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E-mail address: jkawabe@shinshu-u.ac.jp.

[1] Tel./fax: +81 26 269 5562.

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A nonadditive counterpart of weak convergence was studied by Girotto and Holzer in a fairly abstract setting [8]. Among other things, they formulated a nonadditive analog of the portmanteau theorem. This well-known and important theorem allows us to show that weak convergence of nonadditive measures, which is based on the convergence of their Choquet integrals with continuous, bounded, non-negative integrands, coincides with Lévy convergence, which is based on the convergence of nonadditive measures on a special class of measurable sets. Weak convergence of nonadditive measures has been developed further with particular focus on its metrizability and applied to stochastic convergence of measurable mappings defined on a nonadditive measure space [13]. Similar studies can be found in [22] for a possibility measure (called an idempotent probability by the author) and in [11] for a capacity functional of random closed sets.

As aggregation operators with respect to nonadditive measures, there are three major nonlinear integrals; the Choquet integral [4], the Sugeno integral [29], also known as the fuzzy integral [24], and the Shilkret integral [27]. They may be viewed as nonlinear functionals on the product of the space of nonadditive measures and the space of measurable functions. Accordingly, there are three types of weak convergence, that is, weak Choquet convergence, weak Sugeno convergence, and weak Shilkret convergence. At first sight, those integrals may look rather different, because the Choquet integral coincides with the Lebesgue integral for σ-additive measures, while the Sugeno and the Shilkret integrals do not. Thus, different integrals might result in different weak convergences of nonadditive measures.

In this paper, we show that every one of three types of weak convergence of nonadditive measures actually coincides with Lévy convergence and discuss their uniformity on a bounded subset of Lipschitz functions. Our strategy to achieve the purpose is to formulate those results in a unified way for a general nonlinear functional having a perturbative property. This property of a functional is a key concept in this paper and it manages the small change of the functional value by adding a small term to a measure and a measurable function in the domain of the functional [14]. It is worth noting that this unifying approach is also applicable to the Lebesgue integral when the underlying nonadditive measure is σ-additive.

The paper is organized as follows. In Section 2, we recall some definitions on nonadditive measures and nonlinear integrals. In Section 3, in order to examine weak convergence of nonadditive measures in a unified way, we introduce some classes of nonlinear functionals, in particular, the class of perturbative nonlinear functionals. In Section 4, we formulate a type of portmanteau theorem for a perturbative nonlinear integral functional, which yields that Lévy convergence of nonadditive measures coincides with weak convergence based on every one of the Choquet, the Sugeno, and the Shilkret integrals. In Section 5, we discuss the metrizability of weak convergence on the space of all co-continuous regular nonadditive measures on a metric space. In Section 6, we show that weak convergence of nonadditive measures is uniformizable on every bounded set of Lipschitz functions. We apply this uniformity in defining the Fortet–Mourier type metrics on a uniformly equi-autocontinuous set of nonadditive measures. In Section 7, we present conclusions.

2. Preliminaries

In what follows, unless stated otherwise, X is a non-empty set and $\mathcal{A}$ is a field of subsets of X. Let $\mathbb{R}$ and $\mathbb{N}$ denote the set of all real numbers and the set of all natural numbers, respectively. Let $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$ with usual total order. For any $a, b \in \overline{\mathbb{R}}$, let $a \vee b := \max(a, b)$ and $a \wedge b := \min(a, b)$. If $A \subset \mathbb{R}$ is non-empty and not bounded from above (below) in $\mathbb{R}$, we define $\sup A = \infty$ ($\inf A = -\infty$). With this convention, every non-empty subset of $\mathbb{R}$ has a supremum and an infimum in $\overline{\mathbb{R}}$. We adopt the usual conventions for algebraic operations on $\overline{\mathbb{R}}$. We also adopt the convention $(\pm\infty) \cdot 0 = 0 \cdot (\pm\infty) = 0$ and $\inf \emptyset = \infty$.

A function $f : X \to \mathbb{R}$ is said to be *$\mathcal{A}$-measurable* if $\{f \ge t\} := \{x \in X : f(x) \ge t\} \in \mathcal{A}$ and $\{f > t\} := \{x \in X : f(x) > t\} \in \mathcal{A}$ for every $t \in \mathbb{R}$. If f is $\mathcal{A}$-measurable, then so are $f^+ := f \vee 0$ and $f^- := (-f) \vee 0$. Let $\mathcal{F}_b(X)$ denote the set of all bounded, $\mathcal{A}$-measurable functions $f : X \to \mathbb{R}$ with norm $\|f\| := \sup_{x \in X} |f(x)|$ and let $\mathcal{F}_b^+(X) := \{f \in \mathcal{F}_b(X) : f \ge 0\}$. Let χ_A denote the characteristic function of a set A, that is, $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ otherwise.

2.1. Nonadditive measure

A *nonadditive measure* is a set function $\mu : \mathcal{A} \to [0, \infty]$ such that $\mu(\emptyset) = 0$ and $\mu(A) \le \mu(B)$ whenever $A, B \in \mathcal{A}$ and $A \subset B$. It is said to be *finite* if $\mu(X) < \infty$. This type of set function is also called a monotone measure [34],