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Inequalities of the Chebyshev type based on pseudo-integrals

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Abstract

This paper presents the Chebyshev type of pseudo-integral inequalities, with the particular emphasis on the g-semirings. With the suitable choice of the generator g, the pseudo-expectation and pseudo-moments exist for the Cauchy distribution (in general case the standard expectation and moments for the Cauchy distribution are not defined). The computation of the pseudo-expectation for the Cauchy distribution and some examples that illustrate the applicability of the introduced generalization are also given in the paper.

Keywords: Pseudo-operations; Pseudo-additive measure; Pseudo-integral; The Chebyshev type inequality

1. Introduction

The Chebyshev inequality, as an estimation method, has found applications in many different areas that use statistical methods. Aside from the classical engineering examples, the Chebyshev inequality has been used in the computer science in problems related to list decoding, Hadamard code, hashing etc. [11].

Inspired by the nonexistence of the moments for some random variables, resulting in the impossibility of the applications of the integral Chebyshev inequality in these cases, we are going to base the study of the inequalities of the Chebyshev type on the pseudo-integral. A classical example of a random variable for which neither expectation nor moments of order n, $n \in \mathbb{N}$ are defined is a Cauchy random variable. The Cauchy distribution can be introduced as a quotient of two independent normally distributed standardized random variables. It is very important in physics, being a solution of the differential equation which describes the forced resonance. Because of the numerous applications in physics, the Cauchy distribution is also called the Lorentz distribution or the Cauchy–Lorentz distribution. Very useful in solving measurement and calibration problems, it can also be used in risk analysis. Because of its symmetry (being a special case of the Student distribution with one degree of freedom), historically for a short time it was considered as a good candidate for the 'etalon distribution', that role now belonging to the normal distribution. Another interesting

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property of the Cauchy distribution is that its reciprocal is another Cauchy distribution (consequence of its definition as the quotient of two normally distributed random variables). All of this indicates the importance of the Cauchy distribution and opens some possibilities for its applications in different fields.

The replacement of the field of real numbers $\mathbb{R}$ with a closed (semiclosed) interval $[a, b] \subset [-\infty, \infty]$ and the replacement of the operations of addition and multiplication of real numbers with the pseudo-addition and the pseudo-multiplication lead naturally to the pseudo-analysis. In the frame of the pseudo-analysis, three basic classes of semirings are considered. The first and the third class consist of semirings with idempotent pseudo-addition which (under some additional conditions) can be obtained as a limit of a sequence of semirings from the second class, i.e. g-semirings [24]. The significance of the class of g-semirings is therefore evident. Also, the corresponding pseudo-integrals can be obtained as limits of g-integrals [24].

There are many important applications of the pseudo-analysis and the idempotent analysis as its special case. A very useful application is in the field of non-linear partial differential equations [28], object-oriented and generic programming [20,21], game theory [19]) as well as quantum and statistical physics [9]. We can say that there are many important relations and applications to different theoretical and applied areas of purely mathematical and applied mathematical sciences, primarily due to their natural role in expressing and handling various phenomena involving uncertainty [1,18,26,27,29,33].

Non-additive measures and some integrals based on non-additive measures were studied in [5,10,23,24,27]. One of them is introduced by Choquet [10] and has some similar properties as the Lebesgue integral. Another type of integral based on non-additive measures is the Sugeno integral [27,31,33]. The integration of a real-valued function with respect to the $\oplus$-measure is introduced in [25–27].

Furthermore, the integration of a set-valued function has roots in [6]. More about set-valued functions and their integration can be found in [16]. Some definitions and properties of the integral of a set-valued function with respect to the non-additive measure can be found in [17,34,35]. Another interesting study of the pseudo-integral of a set-valued function and the pseudo-integral on an interval-valued function is given in [14,15]. Also, one more type of interval-valued pseudo-integral is studied in [13].

The inequality of the Chebyshev type has been studied in [1,12,13,29,30,32]. The generalizations of different classical integral inequalities in the frame of the pseudo-analysis, such as Minkowski, Hölder, Berwald, Jensen and Cauchy-Schwarz inequalities were studied in [2–4,8,13]. All of these indicate the importance and usefulness of researching this type of inequalities, especially due to the frequent applications in the decision theory, premium principle, probability theory etc.

In the probability theory the concentration results refer to the variety of mathematical estimations that provide the so-called confidence levels, with which a random variable X takes values in certain intervals. Usually those intervals approximate nearness of X to its expectation $E(X)$. Arguably, one of the most important concentration results is the Chebyshev inequality.

The so-called non-centralized form of the Chebyshev inequality is given by

$$P(X \geq \varepsilon) \leq \frac{E(X^2)}{\varepsilon^2},$$

where X is a nonnegative random variable and $\varepsilon > 0$. When the n-th moment $E(X^n)$ exists for the nonnegative random variable X, the Chebyshev inequality has the form

$$P(X \geq \varepsilon) \leq \frac{E(X^n)}{\varepsilon^n}.$$

In particular, for $n = 1$ the previous inequality is also known as the Markov inequality.

The aim of this paper is to present the Chebyshev type of pseudo-integral inequalities, with the particular emphasis on the so-called g-semirings, where the generator g is a monotone continuous function and the operations of pseudo-addition and pseudo-multiplication are given by $x \oplus y = g^{-1}(g(x) + g(y))$ and $x \odot y = g^{-1}(g(x) \cdot g(y))$.

For the Cauchy distribution neither expectation nor the higher moments exist in the real-valued case (see [7]), implying the inapplicability of the Chebishev inequality (or the Markov inequality) on the Cauchy distribution.

We have showed that the Cauchy distribution can be estimated in the pseudo-analysis framework. Namely, with the suitable choice of the generating function g we have showed that the pseudo-expectation and pseudo-moments exist for the Cauchy distribution. We have illustrated the application of the Chebyshev inequality for pseudo-integrals on the Cauchy distribution with a suitable example.