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Tensor products of semilattices and fuzzy ideals

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Abstract

We study (complete) fuzzy ideals on semilattices from the point of view of tensor products of semilattices. We show that the lattice of all complete fuzzy ideals on a semilattice is an extension of tensor product. We define the notion of semicomplete bi-ideals, and show that the lattice of all fuzzy ideals on a distributive lattice is isomorphic to the lattice of all semicomplete bi-ideals. Moreover, we show that the lattice of all (complete) fuzzy ideals on a distributive (algebraic) lattice is distributive.

Keywords: Tensor products; (Complete) fuzzy ideals; Distributive lattices

1. Introduction

Some researchers have studied the modularity of the lattice of all fuzzy ideals of a ring (cf. [4,5,9]). In this paper, we study fuzzy ideals and fuzzy bi-ideals of semilattices or lattices from the point of view of tensor products. Tensor products of semilattices have been studied by using the universal mapping property in [1,2]. In [3], the authors showed that the family of all finitely generated bi-ideals of $\{\vee, 0\}$-semilattices forms a tensor product and studied the structure of the tensor product. In [6], for an arbitrary abstract algebra, the authors defined the notion of weights on the lattice of subuniverses as complete anti-homomorphisms, and showed that the lattice of all fuzzy subalgebras is isomorphic to the lattice of all weights as complete lattices. The notion of weights can be generalized to that of complete fuzzy ideals on complete semilattices. We show that the lattice of all complete fuzzy ideals on a semilattice can be considered as an extension of tensor product of the semilattice and the unit interval $[0, 1]$. From general theory of tensor products of semilattices, we find that the lattice of all (complete) fuzzy ideals on a distributive (algebraic) lattice is distributive.

The paper is organized as follows: Section 2 recalls the notion of bi-ideals of semilattices and the notion of tensor products along lines with [3] and [2]. We introduce the notion of semicomplete bi-ideals, and study the generated semicomplete bi-ideals in the case when one semilattice is $[0, 1]$. In Section 3, we show that the semilattice of all finitely generated fuzzy ideals forms a tensor product of the semilattice and $[0, 1]$. Following Section 3, in Section 4, we study the lattice of all complete fuzzy ideals on an algebraic lattice, and show that the lattice is isomorphic to the

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lattice of all fuzzy ideals on the subsemilattice of all compact elements of the algebraic lattice. We apply the results in this section to the lattice of all fuzzy subalgebras in universal algebra, and explain fuzzy subalgebras from the point of view of tensor products. In Section 5, we show that the lattice of all complete fuzzy ideals can be considered as an extension of tensor product of the semilattice of compact elements and $[0, 1]$. We prove that the lattice of all fuzzy ideals on a distributive lattice is isomorphic to the lattice of all semicomplete bi-ideals, and that the lattice of all (complete) fuzzy ideals on a distributive (algebraic) lattice is distributive.

2. Semicomplete bi-ideals and tensor products

In [3], the authors studied bi-ideals of $\{\vee, 0\}$-semilattices and constructed tensor products. In this section, we define the notion of *semicomplete* bi-ideal, by which we can handle infinitary joins in bi-ideals. Throughout this section, let A, B and C be $\{\vee, 0\}$-semilattices.

Definition 2.1. (Cf. Grätzer–Wehrung [3].) A nonempty subset I of $A \times B$ is a *bi-ideal* of $A \times B$, if it satisfies the following conditions:

(i) I is a hereditary set, that is, if $(a, b) \in I$ and $(x, y) \leq (a, b)$ then $(x, y) \in I$;
(ii) I contains $\perp_{A,B} = (A \times \{0\}) \cup (\{0\} \times B)$;
(iii) I is closed under lateral joins, that is, for $a, a_1, a_2 \in A$ and $b, b_1, b_2 \in B$,
  (a) if $(a_1, b), (a_2, b) \in I$, then $(a_1 \vee a_2, b) \in I$, and
  (b) if $(a, b_1), (a, b_2) \in I$, then $(a, b_1 \vee b_2) \in I$.
  When B is a complete $\{\vee, 0\}$-semilattice, if the bi-ideal I satisfies the following condition (c) instead of (b), we call I *semicomplete* bi-ideal:
  (c) if $\{(a, b_\lambda)\}_{\lambda \in \Lambda} \subseteq I$, then $(a, \vee_{\lambda \in \Lambda} b_\lambda) \in I$.

In [3], the authors noted that the family of all bi-ideals of $A \times B$ forms an algebraic lattice and denoted by $A \bar{\otimes} B$, whose compact elements are just finitely generated bi-ideals. The set of all finitely generated bi-ideals is a $\{\vee, 0\}$-subsemilattice of $A \bar{\otimes} B$ and denoted by $F(A, B)$.

For any subset X of $A \times B$, we denote by $\langle X \rangle$, the bi-ideal generated by X. The bi-ideal $\langle X \rangle$ is given by the following:

Lemma 2.2. *For $X \subseteq A \times B$, let $X_0 = X \cup \perp_{A,B}$, and for every integer $n > 0$, let $X_n = \downarrow \{(x, y) \in A \times B :$ (x, y) is the lateral join of two elements of $X_{n-1}\}$. Then the bi-ideal $\langle X \rangle$ generated by X is equal to $\cup_{n \geq 0} X_n$.*

In the case when B is a complete $\{\vee, 0\}$-semilattice, we denote by $\tilde{F}(A, B)$, the set of all semicomplete bi-ideals of $A \times B$. Since the set $\tilde{F}(A, B)$ is closed under any set intersection, $\tilde{F}(A, B)$ forms a complete lattice and $\{\wedge, 1\}$-subsemilattice of $A \bar{\otimes} B$. By the symbol $[X]$, we denote the semicomplete bi-ideal generated by X. The following lemma is easily seen:

Lemma 2.3. *Let $a, a_1, a_2 \in A$ and $b, b_1 \in A$. Let $\{b_\lambda\}_{\lambda \in \Lambda}$ be a subset of B. Then*

(i) $[(a_1 \vee a_2, b)] = [(a_1, b)] \vee [(a_2, b)]$, *and*
(ii) $[(a, \vee_{\lambda \in \Lambda} b_\lambda)] = \vee_{\lambda \in \Lambda} [(a, b_\lambda)]$.

In the case of $B = [0, 1]$, we would like to capture the semicomplete bi-ideal $[X]$ of $A \times [0, 1]$ by using Lemma 2.2. For $a \in A$, let $Y_a = \{r \in [0, 1] : (a, r) \in \langle X \rangle\}$ and $\partial \langle X \rangle = \{(a, \vee Y_a) : a \in A\}$. Then we have the following proposition.

Proposition 2.4. *Let $X \subseteq A \times [0, 1]$. Then, the semicomplete bi-ideal $[X]$ generated by X is equal to $\langle X \rangle \cup \partial \langle X \rangle$.*

Proof. First, we show that $Y = \langle X \rangle \cup \partial \langle X \rangle$ is a semicomplete bi-ideal of $A \times [0, 1]$.

(i): Let $(a_1, b), (a_2, b) \in Y$. If $(a_1, b), (a_2, b) \in \langle X \rangle$, then obviously $(a_1 \vee a_2, b) \in \langle X \rangle \subseteq Y$. Suppose that $(a_1, b) \in \partial \langle X \rangle$ and $(a_2, b) \in \langle X \rangle$. Since $b = \vee Y_{a_1}$, for an arbitrary $\epsilon > 0$, there exists $b_\epsilon \in [0, 1]$ such that $b - \epsilon < b_\epsilon \leq b$ and $(a_1, b_\epsilon) \in \langle X \rangle$. Then, by the lateral join, $(a_1 \vee a_2, b_\epsilon) \in \langle X \rangle$. Hence $b_\epsilon \leq \vee Y_{a_1 \vee a_2}$. Therefore, $b = \vee_{\epsilon > 0} b_\epsilon \leq \vee Y_{a_1 \vee a_2}$.