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Medians and nullnorms on bounded lattices

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Abstract

Nullnorms are generalizations of triangular norms and triangular conorms with a zero element in the interior of the unit interval. In this paper, we work on nullnorms which are defined on an arbitrary bounded lattice. We introduce a general median-based method for constructing nullnorms by means of triangular norms and triangular conorms. Furthermore, we highlight a significant difference between the (existence and representation of) (idempotent) nullnorms on chains, distributive bounded lattices and general bounded lattices.

Keywords: Nullnorm; Median; Idempotent nullnorm; Distributive bounded lattice; Triangular norm; Triangular conorm

1. Introduction

The history of nullnorms (t-operators) acting on the real unit interval [0, 1] started with the papers “t-operators” (Mas, Mayor and Torrens, 1999 [15]) and “The functional equations of Frank and Alsina for uninorms and nullnorms” (Calvo, De Baets and Fodor, 2001 [4]). These functions are generalizations of triangular norms and triangular conorms. Mas, Mayor and Torrens, in 2002, observed that nullnorms and t-operators are equivalent ([16]). Nullnorms are generalization of triangular norms and triangular conorms with a zero element in the interior of the unit interval and have to satisfy some additional conditions. Note that up to applications in multicriteria decision support, nullnorms are exploited also when defining bipolar integrals, see [8] (Chapter 9).

In this paper, we study nullnorms on bounded lattices. They can serve as pseudo-multiplications when generalizing the concept of the bipolar Sugeno integral to bounded lattices. There are several crucial differences with respect to the standard nullnorms on the real unit interval. For example, for a fixed zero element $a \in L \setminus \{0, 1\}$, there is unique

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idempotent nullnorm in the case when L is a bounded chain (in particular, when L is the real unit interval equipped with the standard ordering of reals). Considering a general bounded lattice $(L, \leq, 0, 1)$, we propose constructions leading in general to different (idempotent) nullnorms. Our constructions generalize the well known fact that each nullnorm on the real unit interval [0, 1] can be represented in the form $V(x, y) = Med(T(x, y), S(x, y), a)$, where a is the zero element, $T : [0, 1]^2 \to [0, 1]$ is a triangular norm and $S : [0, 1]^2 \to [0, 1]$ is a triangular conorm. We aim to investigate this type of construction considering a general bounded lattice.

This paper is organized as follows. In the next section, some preliminaries are given. Our main results are in Section 3 where two construction methods for nullnorms on bounded lattices are introduced and discussed. Finally, some concluding remarks are added.

2. Preliminaries

A bounded lattice $(L, \leq)$ is a lattice which has the top and bottom elements, which are written as 1 and 0, respectively, that is, there exist two elements $1, 0 \in L$ such that $0 \leq x \leq 1$, for all $x \in L$.

Definition 1. (See Birkhoff [3].) Given a bounded lattice $(L, \leq, 0, 1)$ and $a, b \in L$, if a and b are incomparable, in this case we use the notation $a \parallel b$.

Definition 2. (See Birkhoff [3].) Given a bounded lattice $(L, \leq, 0, 1)$ and $a, b \in L$, $a \leq b$, a subinterval $[a, b]$ of L is defined as

$$[a, b] = \{x \in L | a \leq x \leq b\}$$

Similarly, $[a, b) = \{x \in L | a \leq x < b\}$, $(a, b] = \{x \in L | a < x \leq b\}$ and $(a, b) = \{x \in L | a < x < b\}$.

Definition 3. (See [1,7,11,13].) The function $T : L^2 \to L$ $\left(S : L^2 \to L\right)$ is called a triangular norm (triangular conorm) if it is commutative, associative, increasing with respect to both variables and has a neutral element $e = 1$ $(e = 0)$.

The greatest t-norm on $(L, \leq, 0, 1)$ is $T_\wedge$ (inf), the smallest t-conorm is $S_\vee$ (sup), the smallest t-norm is

$$T_W : L^2 \to L, T_W(x, y) = \begin{cases} x \wedge y, & 1 \in \{x, y\} \\ 0, & \text{otherwise} \end{cases}$$

and the greatest t-conorm is

$$S_W : L^2 \to L, S_W(x, y) = \begin{cases} x \vee y, & 0 \in \{x, y\} \\ 1, & \text{otherwise} \end{cases}$$

on $(L, \leq, 0, 1)$.

Definition 4. (See [11,12].) Let $(L, \leq, 0, 1)$ be a bounded lattice.

(i) A t-norm T on L is called $\vee$-*distributive* if for all $x, y, z \in L$

$$T(x, y \vee z) = T(x, y) \vee T(x, z).$$

(ii) A t-conorm S on L is called $\wedge$-*distributive* if for all $x, y, z \in L$

$$S(x, y \wedge z) = S(x, y) \wedge S(x, z).$$

Remark 1. (See [14,9].) The associativity of t-norms allows us to extend each t-norm T in a unique way to an n-ary function in the usual way by induction, defining for each $(x_1, x_2, \ldots, x_n) \in L^n$

$$T_{i=1}^{n} x_i = T\left(T_{i=1}^{n-1} x_i, x_n\right) = T(x_1, x_2, \ldots, x_n).$$