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A new equivalence relation to classify the fuzzy subgroups of finite groups

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Abstract

In this paper a new equivalence relation for classifying the fuzzy subgroups of finite groups is introduced and studied. This generalizes the equivalence relation defined on the lattice of fuzzy subgroups of a finite group that has been used in our previous papers. Explicit formulas for the number of distinct fuzzy subgroups with respect to the new equivalence relation are obtained in some particular cases.

Keywords: Equivalence relations; Fuzzy subgroups; Chains of subgroups; Groups of automorphisms; Group actions; Fixed points; Cyclic groups; Elementary abelian p-groups; Dihedral groups; Symmetric groups

1. Introduction

One of the most important problems of fuzzy group theory is to classify the fuzzy subgroups of a finite group. This topic has enjoyed a rapid development in the last few years. Several papers have treated the particular case of finite abelian groups. Thus, in [8] the number of distinct fuzzy subgroups of a finite cyclic group of square-free order is determined, while [9–11] and [27] deal with this number for cyclic groups of order $p^n q^m$ (p, q primes). Recall here the paper [24] (see also [23]), where a recurrence relation is indicated which can successfully be used to count the number of distinct fuzzy subgroups for two classes of finite abelian groups: cyclic groups and elementary abelian p-groups. The explicit formula obtained for the first class leads in [16] to an important combinatorial result: a precise expression of the well-known central Delannoy numbers in an arbitrary dimension. Next, the study has been extended to some remarkable classes of nonabelian groups: dihedral groups, symmetric groups, finite p-groups having a cyclic maximal subgroup and hamiltonian groups (see [18,20,21] and [25], respectively). The same problems were also investigated for the fuzzy normal subgroups of a finite group (see [22]).

Note that in all our papers mentioned above the fuzzy (normal) subgroups of finite groups have been classified up to the same natural equivalence relation $\sim$ defined on the fuzzy (normal) subgroup lattices. This extends the equivalence

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relation used in Murali's papers [7–11] and gives a powerful connection between the fuzzy subgroups and certain chains of subgroups of finite groups. Recall here the technique initiated in [2] (see also [28]) to derive fuzzy theorems from their crisp versions. Some other different approaches to classify the fuzzy subgroups can be found in [4] and [5].

In the present paper we will treat the problem of classifying the fuzzy subgroups of a finite group G by using a new equivalence relation $\approx$ on the lattice $FL(G)$ of all fuzzy subgroups of G. This is more general than $\sim$, excepting the case when G is cyclic (for which we will prove that $\approx\ =\ \sim$). On the other hand, its definition has a consistent group theoretical foundation, by involving the knowledge of the automorphism group associated to G. In order to count the distinct equivalence classes relative to $\approx$, we shall use an interesting result of combinatorial group theory: the Burnside's lemma (see [13] or [26]). Our method will be exemplified for several remarkable classes of finite groups. Also, we will compare the explicit formulas for the numbers of distinct fuzzy subgroups with respect to $\approx$ with the similar ones obtained in the case of $\sim$. Our approach is motivated by the realization that in a theoretical study of fuzzy groups, fuzzy subgroups are distinguished by their level subgroups and not by their images in $[0, 1]$. Consequently, the study of some equivalence relations between the chains of level subgroups of fuzzy groups is very important. It can also lead to other significant results which are similar with the analogous results in classical group theory.

The paper is organized as follows. In Section 2 we present some preliminary results on the fuzzy subgroups and the group actions of a finite group G. Section 3 deals with a detailed description of the new equivalence relation $\approx$ defined on $FL(G)$ and of the technique that will be used to classify the fuzzy subgroups of G. These are counted in Section 4 for the following classes of finite groups: cyclic groups, elementary abelian p-groups, dihedral groups and symmetric groups. In the final section several conclusions and further research directions are indicated.

Most of our notation is standard and will usually not be repeated here. Basic notions and results on lattices, groups and fuzzy groups can be found in [1,14] and [3] (see also [6]), respectively. For subgroup lattice concepts we refer the reader to [12] and [15].

2. Preliminaries

Let G be a group, $\mathcal{F}(G)$ be the collection of all fuzzy subsets of G and $FL(G)$ be the lattice of fuzzy subgroups of G (see e.g. [6]). The fuzzy (normal) subgroups of G can be classified up to some natural equivalence relations on $\mathcal{F}(G)$. One of them (used in [16–25], as well as in [27]) is defined by

$$\mu \sim \eta \ \text{ iff } \ (\mu(x) > \mu(y) \Longleftrightarrow \eta(x) > \eta(y), \ \text{for all } x, y \in G)$$

and two fuzzy (normal) subgroups μ, η of G are said to be *distinct* if $\mu \nsim \eta$. This equivalence relation generalizes that used in Murali's papers [7–11]. Also, it can be connected to the concept of level subgroup. In this way, suppose that the group G is finite and let $\mu : G \to [0, 1]$ be a fuzzy (normal) subgroup of G. Put $\mu(G) = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ and assume that $\alpha_1 > \alpha_2 > \ldots > \alpha_r$. Then μ determines the following chain of (normal) subgroups of G which ends in G:

$$_\mu G_{\alpha_1} \subset {_\mu G_{\alpha_2}} \subset \ldots \subset {_\mu G_{\alpha_r}} = G. \tag{$*$}$$

A necessary and sufficient condition for two fuzzy (normal) subgroups μ, η of G to be equivalent with respect to $\sim$ has been identified in [27]: $\mu \sim \eta$ if and only if μ and η have the same set of level subgroups, that is they determine the same chain of (normal) subgroups of type $(*)$. This result shows that *there exists a bijection between the equivalence classes of fuzzy (normal) subgroups of G and the set of chains of (normal) subgroups of G which end in G*. So, the problem of counting all distinct fuzzy (normal) subgroups of G can be translated into a combinatorial problem on the subgroup lattice $L(G)$ (or on the normal subgroup lattice $N(G)$) of G: finding the number of all chains of (normal) subgroups of G that terminate in G. Notice also that in our previous papers we have denoted these numbers by $h(G)$ (respectively by $h'(G)$).

Even for some particular classes of finite groups G, as finite abelian groups, the problem of determining $h(G)$ is very difficult. The largest classes of groups for which it was completely solved are constituted by finite cyclic groups (see Corollary 4 of [24]) and by finite elementary abelian p-groups (see the main result of [23]). Recall that if $\mathbb{Z}_n$ is the finite cyclic group of order n and $n = p_1^{m_1} p_2^{m_2} \ldots p_s^{m_s}$ is the decomposition of n as a product of prime factors, then we have