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On non-associative generalizations of MV-algebras and lattice-ordered commutative loops

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Abstract

The aim of the paper is to describe the connections between lattice-ordered commutative loops and certain "basic algebras" which are a non-associative generalization of MV-algebras and are related to commutative semicopulas. This extends the well-known equivalence between lattice-ordered Abelian groups with strong order-unit and MV-algebras.

Keywords: MV-algebra; Semilinear commutative basic algebra; Lattice-ordered commutative loop; Good function

1. Introduction

Various types of binary operations on the real interval [0, 1] extending the classical Boolean conjunction on {0, 1} have been used in order to interpret conjunction in non-classical reasoning. Triangular norms (t-norms) are perhaps the best known example, but, as argued e.g. in [21] or [23], it is often the case that in practical situations the properties of t-norms, and associativity in particular, are unnecessarily restrictive, and hence other, possibly non-associative or non-commutative "conjunctors" have been widely studied, too. After all, the concept of t-norm can be traced back to Menger's paper [27] where associativity was not required.

Our goal is to find a reasonable non-associative generalization of MV-algebras; we want to describe a class of algebras of the same signature as MV-algebras that despite being non-associative would retain most properties of MV-algebras, such as representability by means of subdirect products of linearly ordered algebras. Thus, on the interval [0, 1], the Łukasiewicz t-norm $\odot_Ł$ should be replaced with another "conjunctor" which is non-associative but as close to $\odot_Ł$ as possible. From this point of view, our work is connected with certain (in fact continuous) commutative semicopulas, i.e., binary operations on [0, 1] that are commutative, non-decreasing in each argument and have neutral element 1, which have been originally considered in a statistical context (see e.g. [16,18]). An interesting fact about commutative semicopulas is that with respect to the pointwise order they form a complete lattice which is the Dedekind–MacNeille completion of the poset of t-norms (see [17]).

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There have been a few attempts to define "non-associative MV-algebras" by simply dropping associativity and adding new axioms that allow one to equip the algebras in question with a partial order just as in MV-algebras (see [11] and [24]). Our approach is different and is based on the well-known equivalence between MV-algebras and lattice-ordered Abelian groups with strong order-unit. We first replace groups with (inverse) loops, which seems quite natural when we want to relax associativity on the side of ℓ-groups, and then try to characterize the algebras that are obtained on the side of MV-algebras. To be more specific, given a lattice-ordered commutative (inverse) loop L and a positive element u in L, we make the interval $[0, u]$ of L into a "non-associative MV-algebra", which we denote by $\Gamma(L, u)$, and conversely, we describe those algebras which can be represented as $\Gamma(L, u)$ where, in addition, u is a strong order-unit for L. Of course, in the associative case, this gives precisely Mundici's functor Γ between MV-algebras and lattice-ordered Abelian groups with strong order-unit.

The key property of the Łukasiewicz t-norm $\odot_Ł$ (or the t-conorm $\oplus_Ł$) that we make use of is that it induces antitone involutions on the intervals $[0, a]$ and $[a, 1]$, for every $a \in [0, 1]$. Indeed, the mapping of $x \geq a$ upon $\neg x \oplus_Ł a = 1 + a - x$ is an antitone involution on $[a, 1]$, and dually, the mapping of $x \leq a$ upon $\neg x \odot_Ł a = a - x$ is an antitone involution on $[0, a]$. This observation leads to the so-called "basic algebras", which are algebras $(A, \oplus, \neg, 0, 1)$ of type $(2, 1, 0, 0)$ satisfying certain identities such that the induced poset defined by $x \leq y$ iff $\neg x \oplus y = 1$ is a bounded lattice in which every principal filter $[a, 1]$ has an antitone involution $\gamma_a : x \mapsto \neg x \oplus a$, or dually, every principal ideal $[0, a]$ has an antitone involution $\delta_a : x \mapsto \neg(x \oplus \neg a)$. The exact definition is given in Section 2. These algebras were introduced in [8]; the motivation was to generalize MV-algebras in the same way in which orthomodular lattices generalize Boolean algebras, and at the same time, to generalize orthomodular lattices in the same way in which MV-algebras generalize Boolean algebras. This approach provided a new perspective on lattice effect algebras which have been studied in the context of quantum structures (see e.g. [19]). The smallest variety containing both MV-algebras and orthomodular lattices was recently axiomatized in [26].

We should make a comment on terminology here. The name "basic algebra" is by no means good and it was only meant to indicate that "basic algebras" were a common framework for MV-algebras, orthomodular lattices and other structures considered in [8], and we continue to use it because we actually don't have anything better. Besides, other possible names (such as NMV-algebras [11] or WMV-algebras [24]) have already been used in the literature and we believe that "MV" should be reserved from algebras that are closer to MV-algebras than "basic algebras" as such.

The paper is organized as follows. In Section 2, we recall all relevant facts about basic algebras, especially about monotone and commutative ones, where "monotone" means that the addition $\oplus$ non-decreasing in both arguments. Section 3 is devoted to basic algebras on the intervals $[0, u]$ in the positive cones of lattice-ordered commutative loops. It is shown that for any lattice-ordered commutative loop L, the interval $[0, u]$ can be made into a monotone basic algebra $\Gamma(L, u)$ which, however, need not be commutative. We find a condition under which $\Gamma(L, u)$ is commutative. In Section 4, generalizing Chang's construction (see [14]), we prove that every linearly ordered commutative basic algebra is of the form $\Gamma(L, u)$ for a suitable linearly ordered commutative inverse loop and a strong order-unit u in L. In Section 5 we prove that every commutative basic algebra which is a subdirect product of linearly ordered factors (we call such algebras "semilinear") is isomorphic to some $\Gamma(L, u)$ where again u is a strong order-unit for L. Our method is based on Mundici's good sequences (see [15,22,28]), except that similarly to [1] we actually work with "good functions", which are certain mapping from $\mathbb{Z}$ to the algebra. This directly gives L, while the classical construction leads to the positive cone of L from which L must be constructed subsequently. The final Section 6 contains some remarks on lexicographic products and we present a new type of examples of proper (non-associative) commutative basic algebras.

2. Preliminaries to "basic algebras"

As we have already mentioned, these algebras were introduced in [8] as a common generalization of MV-algebras and orthomodular lattices. The basic observation behind the definition is that both MV-algebras and orthomodular lattices are bounded lattices where all principal filters as well as all principal ideals bear certain antitone involutions. A *lattice with (sectional) antitone involutions* was defined as a bounded lattice $(A, \vee, \wedge, 0, 1)$ every principal filter $[a, 1]$ of which is equipped with a fixed antitone involution γ_a, and the corresponding *basic algebra* $(A, \oplus, \neg, 0, 1)$ was defined by setting $\neg x = \gamma_0(x)$ and $x \oplus y = \gamma_y(\neg x \vee y)$. It is evident that once we are given antitone involutions on all principal filters, we are also given antitone involutions on all principal ideals, and vice versa; indeed, if φ_a are antitone involutions on the principal filters $[a, 1]$ (respectively, on the principal ideals $[0, a]$) of A, and if $\neg$ denotes