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General approximation of fuzzy numbers by F-transform

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Abstract

In this paper we will prove that in most of the cases the extended inverse fuzzy transform preserves the quasi-concavity of a fuzzy number and hence it can be used to generate fuzzy numbers by approximating the restriction of the membership function to its support. In the case of continuous fuzzy numbers with cores containing more than one element, the rate of uniform convergence is of linear type and the same holds when we approximate the important characteristics of a fuzzy number such as the value or the ambiguity. Moreover we have the preservation of the support and the convergence of the core which in addition can be determined precisely. In the case of continuous fuzzy numbers with one-element core, it is in general necessary to normalize the approximation, but the support is preserved again and the core can be determined exactly in this case too. Moreover, the approximations have again linear rate of uniform convergence.

Keywords: Fuzzy number; F-transform; Ambiguity; Value; Quasi-concavity

1. Introduction

Quite often fuzzy numbers are difficult to handle due to a too complex representation. As an alternative, we can simplify the representation of a fuzzy number by using suitable approximations. This topic has brought the attention of many researchers in the last few years. A very important issue is the approximation of general fuzzy numbers by other fuzzy numbers with simpler form (e.g. triangular, trapezoidal, parametric LR or LU, and others) with respect to some metric (Hausdorff, L_2-type, ...). Here we can distinguish approximations without other restrictions (see e.g. [1,10,17,24]) and approximations with additional requirements such as the preservation of the expected interval, core, ambiguity or value (see e.g. [2–4,15,21]). A more detailed discussion regarding the state of the art in this topic can be found in paper [8]. Obviously, such approximation suffices in applications which need a certain characteristic of the

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fuzzy number, for example its expected interval. However, as in the classical topic of the approximation of functions, sometimes, there exist concrete problems where we need to preserve as much as possible the information carried out by a fuzzy number (see also paper [9] where this shortcoming is mentioned too). When we use a trapezoidal approximation there is a limited number of characteristics that can be preserved because a trapezoid depends on four parameters only. Therefore, one of the best methods to preserve multiple characteristics (or to have a convergent process) is to use sequences of fuzzy numbers. In this way we can also obtain the important property of convergence with respect to the uniform norm.

In this paper our main result is to find sequences of fuzzy numbers which converge uniformly to a given fuzzy number and such that we have a convergent process with respect to the important characteristics. We will see that, by using the inverse F-transform, introduced in [18] (see also [6] for its approximation properties), we can generate sequences with desired properties, such as invariance of support, core, ambiguity, value and expected interval. The approximation of general fuzzy numbers by sequences of fuzzy numbers is another topic which interested researchers in fuzzy analysis. In the book of Diamond and Kloeden [12, pp. 124–126] the Bernstein type operators are attached to convex fuzzy subsets of $\mathbb{R}^d$. These approximations preserve the support and the quasi-concavity but in order to generate fuzzy numbers, in general, we always need to normalize the Bernstein approximation. Another problem appears when the core of the fuzzy number has more than one element (such fuzzy numbers are often called fuzzy intervals) because the core of its Bernstein approximation is reduced to a single element and hence we do not have convergence with respect to the core (see [9]). This may lead to worse approximations of the ambiguity and value, comparing with approximations where we have convergence with respect to the core. Such qualitative approximations can be obtained by using the inverse F-transform approach. More precisely, in this paper, we will approximate fuzzy numbers by using the extended F-transform setting proposed in [19] and we will see that the extended inverse F-transform of a fuzzy number will always preserve the support and the quasi-concavity property. For the case of fuzzy intervals we will not need to normalize the approximation when we have sufficient knots on the decomposition. In addition, the core of the approximation converges to the core of the fuzzy number and it can be determined precisely. Moreover, for a fuzzy interval u having a continuous membership function, the core of the approximation is included in the core of u. When the core of the fuzzy number has one element (quite often such fuzzy numbers are called unimodal) then we have the same properties with the only difference that we need to normalize the approximation. Although in general this is a drawback especially considering computer implementation, it is not the case when we consider the F-transform setting because we can always determine exactly the maximum value of the approximation. Another advantage comparing with other approaches (including the before mentioned Bernstein approximation approach) is that we have a better, linear rate of uniform convergence meaning that we can obtain a desired error of approximation with less computational complexity and this holds for the approximations of the ambiguity and value as well.

The paper is organized as follows. Section 2 contains some basics about fuzzy numbers and their important characteristics. In Section 3 we recall some important facts about the direct F-transform and the inverse F-transform with a special interest on the extended direct and inverse F-transform introduced in [19] (to obtain the preservation of monotonicity). In Section 4 it is proved that the inverse F-transform (with respect to a uniform partition) of a continuous fuzzy number with core having more than one element preserves the quasi-concavity (Theorem 12). Therefore, by using the extended inverse F-transform we can generate fuzzy numbers which have good approximation and shape preserving properties (Theorem 14). More exactly, for an h-uniform partition of the support of fuzzy number u we obtain the order of uniform approximation $\omega(u, h)$. Interestingly, if the fuzzy number has one-element core then by so-called regular uniform partitions (see Definition 15), again we obtain the preservation of the quasi-concavity of the extended inverse F-transform (Theorem 16). In Section 5 we investigate the approximation of the value and ambiguity. As a result of general interest, we express these characteristics in terms of the membership function (Theorem 20). In the case when a fuzzy number v approximates u with respect to the Chebyshev–Hamming metric D_C, we estimate the error of approximation of the ambiguity and value (Theorem 22). All these results can be used to approximate the value and the ambiguity of a fuzzy number by using the value and ambiguity of the extended inverse F-transform or normalized extended inverse F-transform (see Theorem 27 and Theorem 29, respectively). In Section 6 we test the theoretical results on examples. The paper ends with some concluding remarks where the main results are summarized and further research is proposed.